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D-term cosmic strings from $N = 2$ supergravity

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Abstract: We describe new half-BPS cosmic string solutions in $N = 2$, $d = 4$ supergravity coupled to one vector multiplet and one hypermultiplet. They are closely related to D-term strings in $N = 1$ supergravity. Fields of the $N = 2$ theory that are frozen in the solution contribute to the triplet moment map of the quaternionic isometries and leave their trace in $N = 1$ as a constant Fayet-Iliopoulos term. The choice of U(1) gauging and of special geometry are crucial. The construction gives rise to a non-minimal Kähler potential and can be generalized to higher dimensional quaternionic-Kähler manifolds.

Keywords: Supergravity Models, Supersymmetry Breaking, Extended Supersymmetry.

Contents

1. Introduction

In this paper we discuss the classical embedding of D-term string solutions of $d = 4$, $N = 1$ supergravity into $N = 2$ theories. D-term strings in supergravity [1, 2] are BPS-solutions of the supersymmetric Einstein-Higgs Abelian gauge field model coupled to supergravity, with half of the supersymmetries unbroken, saturating the Bogomol'nyi bound. Earlier work on BPS-strings and the Bogomol'nyi bound in $d = 3$ supergravity can be found in [3 − 5]. Further work on the properties of D-term strings includes the analysis of zero modes [6 − 8] and BPS axionic strings [9 − 11]. D-term strings are expected to form after D-term inflation [12]; a recent assessment of their impact on the Cosmic Microwave Background can be found in [13] and references therein. The supergravity model considered in [1] contains one vector-multiplet, one chiral multiplet and the graviton multiplet; the model has a complex scalar, charged under U(1), that parametrizes the trivial (flat) internal Kähler-Hodge space with Kähler potential $\mathcal{K} = \phi\phi^*$, a D-term potential, and a vanishing superpotential. Essential for the construction is the constant *Fayet-Iliopoulos term* (henceforth denoted FI term [14]) appearing in the D-term potential; issues concerning this term are clarified in [15, 16].

Engineering such a constant FI term from $N = 2$ supergravity is not trivial: an FI term in $N = 2$ supergravity corresponds to the case of an arbitrary constant in the *moment map*, which can only occur when there are no hypermultiplets. However, one needs at least one hypermultiplet to play the role of matter charged under the U(1) gauging that we intend to perform (scalars of vector multiplets cannot be charged under an Abelian gauging). To perform the gauging, one chooses a suitable isometry of the (quaternionic-Kähler) hypermultiplet target space. The moment map corresponding to the isometry is then dependent on some or all of the hypermultiplet scalars (see [17] for a review). In this paper we propose an ansatz that circumvents this problem. Formally, the ansatz is closely related to a *consistent truncation* of $N = 2$ to $N = 1$, such that the cosmic string solutions would be half-BPS D-term strings in the $N = 1$ theory, with (constant) FI term and vanishing superpotential. The interesting point here is that we show they are also half-BPS in the $N = 2$ theory.

The authors of [1] recently conjectured that D-term strings might be the low energy manifestation of D1 or wrapped Dp branes (see [18, 19] and the recent review [20]), based on the observation that D-term strings are the only half-BPS vortices available in $N = 1$ supergravity. Support for this conjecture was provided in [21 − 23]. Since then, it has been shown that other D-term BPS vortices (axisymmetric solutions) exist in $N = 1$ supergravity (for instance semilocal "vortices" in the Bogomol'nyi limit [24, 25], which have arbitrarily wide cores, see also [26]), an observation that may be relevant to the conjecture. In any case, the idea of matching the BPS D-brane states of superstring theory with some low-energy counterparts in supergravity is a very interesting one. If such a correspondence can be made, then one would in principle expect to find half-BPS vortices also in other low-energy manifestations, in particular in compactifications with $N = 2$. Our results show that this expectation is correct.

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Consistent truncations of $N = 2$ to $N = 1$ are described in [27, 28], where the authors consider the truncated $N = 2$ BPS equations in the reduced $N = 1$ theory and analyse the resulting geometric conditions on the special Kähler and quaternionic-Kähler target spaces. We rely heavily on these results to show the consistency of the ansatz. The other ingredients are the choice of U(1)-isometry to be gauged and the choice of special geometry, which are essential to the construction. We give two explicit examples of "minimal" $N = 2$ models with one vector- and one hypermultiplet. They have special Kähler space $\mathrm{SU}(1,1)/\mathrm{U}(1)$ and a symmetric one-dimensional quaternionic-Kähler space, for which there are two choices, both reducing after truncation to the same $N = 1$ action (up to a normalization) with one chiral and one vector multiplet. We also comment on the embedding in models with more than one hypermultiplet.

The $N = 1$ action resulting from the truncation has a complex scalar that parametrizes the Kähler-Hodge manifold $\mathrm{SU}(1,1)/\mathrm{U}(1)$. We solve the BPS equations and we describe the cosmic string half-BPS solution, in close analogy to the results in [1]. As an aside, we use a Bogomol'nyi-type argument to prove that cylindrically symmetric $N = 1$ D-term strings with complex scalars parametrizing any Kähler-Hodge manifold, and charged under an Abelian U(1), have a mass per unit length that is bounded below by the Gibbons Hawking surface term, and that the bound is attained by the BPS solutions (if these exist). Since we have assumed a specific ansatz, we cannot immediately conclude that the solutions are stable but at least they minimize the energy within this class of configurations.

We begin in section 2 by reviewing some basic ingredients of $N = 2$ supergravity that we will use, see also [17]. It includes the kinetic terms of the Lagrangian and the potential, as well as the leading terms of the supersymmetry transformations. We remind the reader of the problem concerning constant FI terms in $N = 2$ supergravity in section 2.2. The mechanism by which FI terms in $N = 1$ originate from $N = 2$ theories, and its relation to consistent truncations, is explained in section 3. It is illustrated there on the 1-dimensional quaternionic-Kähler symmetric spaces. In section 4, we first define the special geometry that is used in the models that we consider. A field configuration is then presented with the property that the bosonic action becomes just the one of an $N = 1$ matter-coupled supergravity system, and that the BPS equations decouple. This resulting system is studied further in section 5, where we introduce an ansatz for a cosmic string. Two different parametrizations of the 1-dimensional complex projective space are convenient in different settings. We compare them, and after splitting the conditions, the BPS equations can be solved. From the $N = 2$ point of view, the relation between the half-plane and the unit disk is an SU(2) rotation of the quaternionic structure. In the $N = 1$ reduced theory, such an SU(2)-connection is seen as a Kähler transformation or better as a transformation of the Kähler U(1)-connection. As this is an example of D-term strings with non trivial Kähler-Hodge target spaces (see also [11]), we give in section 6 some general remarks on such $N = 1$ string solutions. Section 7 gives our conclusions.

We have included an appendix A with notations, and an appendix B to expose the parametrizations that we use for the coset spaces.

vielbein	e_μ^a	$\mu, a = 0, \ldots, 3$
gravitini	ψ_μ^i, $\psi_{\mu i}$	$i = 1, 2$
vectors	W_μ^I	$I = 0, \ldots, n_V$
gaugini	λ_i^α, $\lambda^{\bar\alpha i}$	$\alpha = 1, \ldots, n_V$
hyperini	ζ^A , ζ_A	$A = 1, \ldots, 2n_H$
Kähler manifold scalars	z^α, $\bar z^{\bar\alpha}$	
hyperscalars	q^X	$X = 1, \ldots, 4n_H$

Table 1: Multiplets and fields of the super-Poincaré theories.

2. Basic formulae of matter-coupled $N = 2$ supergravity

2.1 Fields and kinetic terms

We repeat here the basic formulae of $N = 2$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets [29 − 31, 17], though in the following sections we will mostly use $n_V = n_H = 1$. These theories contain the fields given in table 1. For the fermions, we write the left-handed components [projected by $\frac{1}{2}(1+\gamma_5)$] on the left-hand side and the right-handed component on the right-hand side. The vector multiplets contain the complex[1] scalars z^α, while the scalars for the hypermultiplets are written here as real fields q^X. The leading (kinetic) terms of the action are then

$$e^{-1}\mathcal{L}_{\rm kin} = \frac{1}{2}R - \bar\psi_\mu^i\gamma^{\mu\nu\rho}\nabla_\nu\psi_{\rho i} + \frac{1}{2}\,{\rm Im}\left(\mathcal{N}_{IJ}F_{\mu\nu}^{+I}F^{+\mu\nu J}\right) - \frac{1}{2}g_{\alpha\bar\beta}\bar\lambda_i^\alpha\not\nabla\lambda^{\bar\beta i} - g_{\alpha\bar\beta}\nabla_\mu z^\alpha\nabla^\mu\bar z^{\bar\beta}$$
$$-\frac{1}{2}g_{XY}\nabla_\mu q^X\nabla^\mu q^Y - 2\bar\zeta^A\not D\zeta_A. \qquad (2.1)$$

See Appendix A for metric and spinor conventions and $e = \det(e_\mu^a)$. The derivatives ∇ are in this approximation ordinary spacetime derivatives, but are in the full theory covariant derivatives that we will explain below, see (2.23). Here $F_{\mu\nu}^{+I}$ is the self-dual combination

$$F_{\mu\nu}^{\pm I} = \frac{1}{2}\left(F_{\mu\nu}^I \mp \frac{1}{2}{\rm i}e\varepsilon_{\mu\nu\rho\sigma}F^{I\rho\sigma}\right), \qquad F_{\mu\nu}^I = 2\partial_{[\mu}W_{\nu]}^I = \partial_\mu W_\nu^I - \partial_\nu W_\mu^I, \qquad (2.2)$$

where we restrict ourselves to the Abelian case. The quantities $\mathcal{N}_{IJ}$, $g_{\alpha\bar\beta}$ and g_{XY} are related to the chosen geometry for the vector multiplets and the hypermultiplets, which we will now describe.

The vector multiplet action describes a special Kähler manifold. Everything is determined in terms of a prepotential[2], $F(Z)$, which should be holomorphic and homogeneous of second degree in variables Z^I. The basic object is a $2(n_V + 1)$-component symplectic section

$$V(z,\bar z) = \begin{pmatrix} X^I \\ M_I \end{pmatrix} = {\rm e}^{\mathcal{K}(z,\bar z)/2}v(z), \qquad v(z) = \begin{pmatrix} Z^I(z) \\ F_I \end{pmatrix},$$

[1] For the quantities in the Kähler manifolds, we use the bar notation for complex conjugation. In the hypermultiplets we distinguish hermitian conjugation indicated by the bar, and complex conjugation indicated by ∗. Charge conjugation, which is complex conjugation for bosons, replaces left-handed fermions with right-handed ones.

[2] A more general description without a prepotential is possible [32, 33], but is not needed here.

$$M_I = \frac{\partial}{\partial X^I} F(X), \qquad F_I = \frac{\partial}{\partial Z^I} F(Z). \tag{2.3}$$

Here $Z^I(z)$ are arbitrary functions (up to conditions for non-degeneracy), reflecting the freedom of choice of coordinates z^α. The lower components depend on the prepotential. A constraint in terms of a symplectic inner product

$$< V, \bar{V} >= X^I \bar{M}_I - M_I \bar{X}^I = \mathrm{i}, \tag{2.4}$$

determines the Kähler potential as

$$\mathrm{e}^{-\mathcal{K}(z,\bar{z})} = -\mathrm{i}\langle v, \bar{v}\rangle = -\mathrm{i} Z^I \bar{F}_I + \mathrm{i} F_I \bar{Z}^I. \tag{2.5}$$

The metrics for the scalars and for the vectors are then determined by

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}(z,\bar{z}) = \mathrm{i}\langle \mathcal{D}_\alpha v, \mathcal{D}_{\bar{\beta}} \bar{v}\rangle, \qquad \mathcal{N}_{IJ} \equiv \Big(F_I \;\; \bar{\mathcal{D}}_{\bar{\alpha}} \bar{F}_I \Big) \Big(Z^J \;\; \bar{\mathcal{D}}_{\bar{\alpha}} \bar{Z}^J \Big)^{-1}, \tag{2.6}$$

where covariant derivatives are defined by

$$\mathcal{D}_\alpha v = \partial_\alpha v + (\partial_\alpha \mathcal{K}) v, \qquad \mathcal{D}_{\bar{\alpha}} \bar{v} = \partial_{\bar{\alpha}} \bar{v} + (\partial_{\bar{\alpha}} \mathcal{K}) \bar{v}. \tag{2.7}$$

Due to the presence of the prepotential, one can give also the expressions

$$\begin{aligned} \mathrm{e}^{-\mathcal{K}} &= -Z^I N_{IJ} \bar{Z}^J, \qquad \mathcal{N}_{IJ} = \bar{F}_{IJ} + \mathrm{i} \frac{N_{IN} N_{JK}\, Z^N Z^K}{N_{LM}\; Z^L Z^M}, \\ N_{IJ} &\equiv 2\,\mathrm{Im}\, F_{IJ} = -\mathrm{i} F_{IJ} + \mathrm{i} \bar{F}_{IJ}, \qquad F_{IJ} = \frac{\partial^2}{\partial Z^I \partial Z^J} F(Z). \end{aligned} \tag{2.8}$$

A useful relation between the two metrics is

$$-\frac{1}{2} (\mathrm{Im}\, \mathcal{N})^{-1|IJ} = \mathcal{D}_\alpha X^I g^{\alpha\bar{\beta}} \mathcal{D}_{\bar{\beta}} \bar{X}^J + \bar{X}^I X^J. \tag{2.9}$$

The hypermultiplets describe a quaternionic-Kähler manifold. The starting point for a supergravity description is the vielbein f_X^{iA}. We need furthermore a symplectic metric $\mathbb{C}_{AB}$, antisymmetric and with complex conjugate $\mathbb{C}^{AB}$, such that it satisfies the same relation as ε_{ij}:

$$\mathbb{C}_{AC} \mathbb{C}^{BC} = \delta_A^B. \tag{2.10}$$

The vielbein satisfies the reality property

$$\left(f_X^{iA}\right)^* = f_{XiA} = f_X^{jB} \varepsilon_{ji} \mathbb{C}_{BA}. \tag{2.11}$$

The inverse of the vielbein as $4n_H \times 4n_H$ matrix is f_{iA}^X. The vielbein determines the metric and the quaternionic structures:

$$2 f_X^{jA} f_{YiA} = \delta_i^j g_{XY} + J_{XYi}{}^j, \qquad f_{iA}^X = g^{XY} f_{YiA}. \tag{2.12}$$

$J_{XYi}{}^j$ is traceless in the i,j indices, and is decomposed in the 3 complex structures as

$$J_{XYi}{}^j = \mathrm{i} (\sigma^x)_i{}^j J_{XY}^x, \qquad x = 1,2,3, \tag{2.13}$$

where σ^x are the Pauli matrices. These complex structures are covariantly constant using the Levi-Civita connection $\nabla^{\rm LC}$ and an SU(2) connection ω_X^x as

$$\nabla_X J^x_{YZ} \equiv \nabla^{\rm LC}_X J^x_{YZ} + 2\varepsilon^{xyz}\omega^y_X J^z_{YZ} = 0. \tag{2.14}$$

The complex structure is proportional to the SU(2) curvature. Written as forms, this relation is

$$\mathcal{R}^x \equiv {\rm d}\omega^x + \varepsilon^{xyz}\omega^y\omega^z = \frac{1}{2}\nu J^x, \qquad \nu = -\kappa^2 = -1. \tag{2.15}$$

The value of ν is arbitrary in quaternionic-Kähler manifolds, but invariance of the action in supergravity relates it to the gravitational coupling constant, which we have put equal to 1 in this paper.

2.2 Isometries and the moment map

As we will consider only an Abelian vector multiplet, we will restrict this presentation to the gauging of isometries in the hypermultiplet. We consider the transformation with parameters α^Λ:

$$\delta_G q^X = -g\alpha^\Lambda k_\Lambda^X, \tag{2.16}$$

and k_Λ^X are Killing vectors. In a quaternionic-Kähler manifold any isometry normalizes[3] the quaternionic structure [34]. The Killing vector can then be derived from a triplet moment map $\mathcal{P}^x_\Lambda$:

$$\iota_\Lambda J^x \equiv k_\Lambda^X J^x_{XY}{\rm d}q^Y = 2\nabla\mathcal{P}^x_\Lambda \equiv 2({\rm d}\mathcal{P}^x_\Lambda + 2\varepsilon^{xyz}\omega^y\mathcal{P}^z_\Lambda) \qquad \text{or} \qquad 4n_H\mathcal{P}^x_\Lambda = J^{xXY}\partial_X k_{Y\Lambda}. \tag{2.17}$$

Due to the non-trivial SU(2) connection, the triplet moment maps cannot be shifted by arbitrary constants, (unlike in rigid $N=2$ supersymmetry, where only ${\rm d}\mathcal{P}^x$ occurs in $\iota_\Lambda J^x$). These constants would be the FI terms, and for the above reasons their introduction in $N=2$ supergravity is problematic.

The moment map can also be described in another way. A Killing vector preserves the connection ω^x and Kähler two forms J^x only modulo an SU(2) rotation. Denoting by $\mathcal{L}_\Lambda$ a Lie derivative with respect to k_Λ, we have

$$\mathcal{L}_\Lambda\omega^x = -\frac{1}{2}\nabla r^x_\Lambda, \quad \mathcal{L}_\Lambda J^x = \varepsilon^{xyz}r^y_\Lambda J^z, \tag{2.18}$$

where r^x_Λ is known as an SU(2) *compensator*. The SU(2)-bundle of a quaternionic manifold is non-trivial and therefore it is impossible to get rid of the compensator r^x_Λ by a redefinition of the SU(2) connections.[4] The moment map can be expressed in terms of the triplet of connections ω^x and the compensator r^x_Λ in the following way [35]:

$$\mathcal{P}^x_\Lambda = \frac{1}{2}r^x_\Lambda + \iota_\Lambda\omega^x. \tag{2.19}$$

[3] This means that the Lie derivative of the three complex structures is a linear combination of the complex structures themselves.

[4] Again, this is in contrast with $N=2$ rigid supersymmetry, since hyper-Kähler manifolds have a trivial SU(2) bundle, and therefore no compensator.

2.3 Gauging and supersymmetry transformations

We now gauge some of the isometries mentioned above, and connect them to gauge transformations of the vectors, such that the indices Λ are replaced by I with

$$\delta_G W_\mu^I = \partial_\mu \alpha^I. \tag{2.20}$$

This modifies the supersymmetry transformation laws. The normalizations are such that the bosons transform as

$$\begin{aligned}
\delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}^i \gamma^a \psi_{\mu i} + \frac{1}{2}\bar{\epsilon}_i \gamma^a \psi_\mu^i, \\
\delta W_\mu^I &= \frac{1}{2}(\mathcal{D}_\alpha X^I)\varepsilon^{ij}\bar{\epsilon}_i \gamma_\mu \lambda_j^\alpha + \frac{1}{2}(\mathcal{D}_{\bar{\alpha}} \bar{X}^I)\varepsilon_{ij}\bar{\epsilon}^i \gamma_\mu \lambda^{\bar{\alpha} j} + \varepsilon^{ij}\bar{\epsilon}_i \psi_{\mu j} X^I + \varepsilon_{ij}\bar{\epsilon}^i \psi_\mu^j \bar{X}^I, \\
\delta z^\alpha &= \frac{1}{2}\bar{\epsilon}^i \lambda_i^\alpha, \\
\delta q^X &= -\mathrm{i} f_{iA}^X \bar{\epsilon}^i \zeta^A + \mathrm{i} f^{XiA}\bar{\epsilon}_i \zeta_A.
\end{aligned} \tag{2.21}$$

For a bosonic configuration, the $N = 2$ supersymmetry transformations of the left-handed fermionic fields are (see appendix A for a description of our conventions):

$$\begin{aligned}
\delta \psi_\mu^i &= \nabla_\mu(\omega)\epsilon^i - g\gamma_\mu S^{ij}\epsilon_j + \frac{1}{4}\gamma^{\rho\sigma} T_{\rho\sigma}^- \varepsilon^{ij}\gamma_\mu \epsilon_j, \\
\delta \lambda_i^\alpha &= \nabla\!\!\!\!/\, z^\alpha \epsilon_i - \frac{1}{2} g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{X}^I \,\mathrm{Im}\,\mathcal{N}_{IJ} F_{\mu\nu}^{-J}\gamma^{\mu\nu}\varepsilon_{ij}\epsilon^j + g N_{ij}^\alpha \epsilon^j, \\
\delta \zeta^A &= \frac{1}{2}\mathrm{i} f_X^{Ai} \nabla\!\!\!\!/\, q^X \epsilon_i + g\mathcal{N}^{iA}\varepsilon_{ij}\epsilon^j.
\end{aligned} \tag{2.22}$$

The covariant derivatives are

$$\begin{aligned}
\nabla_\mu(\omega)\epsilon^i &\equiv \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\epsilon^i + \frac{1}{2}\mathrm{i} A_\mu \epsilon^i + V_{\mu j}{}^i \epsilon^j, \\
\nabla_\mu z^\alpha &= \partial_\mu z^\alpha + g W_\mu^I k_I^\alpha, \\
\nabla_\mu q^X &= \partial_\mu q^X + g W_\mu^I k_I^X.
\end{aligned} \tag{2.23}$$

We included here the effect of a gauging in the vector multiplet sector by the Killing vector k_I^α describing the transformations under the gauge symmetry of the vector multiplet scalar similar to the definition of k_I^X as in (2.16) for the hypermultiplet scalars. The SU(2) connection $V_{\mu i}{}^j$ is related to the quaternionic-Kähler $SU(2)$:

$$V_{\mu i}{}^j = \partial_\mu q^X \omega_{Xi}{}^j + g W_\mu^I \mathcal{P}_{Ii}{}^j. \tag{2.24}$$

A_μ are the components of the one-form gauge field of the Kähler U(1):

$$A = -\frac{1}{2}\mathrm{i}\left(\partial_\alpha \mathcal{K}\mathrm{d}z^\alpha - \partial_{\bar{\alpha}}\mathcal{K}\mathrm{d}\bar{z}^{\bar{\alpha}}\right). \tag{2.25}$$

In the case of gauging in the vector multiplet sector, this is modified by a scalar moment map similar to the SU(2) connection. The dressed graviphoton is given by

$$T_{\mu\nu}^- = F_{\mu\nu}^{-I}\,\mathrm{Im}\,\mathcal{N}_{IJ} X^J. \tag{2.26}$$

The fermionic shifts (mass matrices) are given in terms of the prepotentials and Killing vectors of the quaternionic-Kähler geometry (dressed with special geometry data) as follows:

$$
\begin{aligned}
S^{ij} &\equiv -\mathcal{P}_I^{ij} X^I , \\
N_{ij}^{\alpha} &\equiv \varepsilon_{ij} k_I^{\alpha} \bar{X}^I - 2\mathcal{P}_{Iij} \bar{\mathcal{D}}_{\bar{\beta}} \bar{X}^I g^{\alpha\bar{\beta}} , \qquad \mathcal{N}^{iA} \equiv -\mathrm{i} f_X^{iA} k_I^X \bar{X}^I .
\end{aligned}
\tag{2.27}
$$

They determine also the potential by

$$
\begin{aligned}
g^{-2}\mathcal{V} &= -6 S^{ij} S_{ij} + \frac{1}{2} g_{\alpha\bar{\beta}} N_{ij}^{\alpha} N^{\bar{\beta} ij} + 2\mathcal{N}^{iA}\mathcal{N}_{iA} \\
&= 4\left(U^{IJ} - 3\bar{X}^I X^J\right)\mathcal{P}_I^x \mathcal{P}_J^x + \left(g_{\alpha\bar{\beta}} k_I^{\alpha} k_J^{\bar{\beta}} + 2 g_{XY} k_I^X k_J^Y\right)\bar{X}^I X^J ,
\end{aligned}
\tag{2.28}
$$

where

$$
U^{IJ} \equiv g^{\alpha\bar{\beta}} \mathcal{D}_\alpha X^I \mathcal{D}_{\bar{\beta}} \overline{X}^J = -\frac{1}{2}\left(\mathrm{Im}\mathcal{N}\right)^{-1|IJ} - \overline{X}^I X^J . \tag{2.29}
$$

For the minimal models described in this work we will consider the case of one vector multiplet, of which the scalar z parametrizes $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$, and one hypermultiplet. A $\mathrm{U}(1)$ isometry of the quaternionic-Kähler space with associated Killing vector k_1 will be gauged with the vector W^1.

3. FI terms from truncations of $N = 2$ to $N = 1$

3.1 General method

If a reduction of $N = 2$ to $N = 1$ is performed by consistently truncating the second gravitino ψ_μ^2, the $N = 1$ superpotential is a function of $\mathcal{P}^1 + \mathrm{i}\mathcal{P}^2$ and the D-term is constructed out of $\mathcal{P}^3$ [27].

The aim now is to find a gauging that is consistent with the truncation to $N = 1$. The gauging should give rise to a moment map with $\mathcal{P}_1 = \mathcal{P}_2 = 0$ after truncation, and a non-zero component $\mathcal{P}_3$ that will result in a D-term potential. Thus, $\mathcal{P}_3$ should contain a term that acts as a FI term in the resulting $N = 1$ theory. *We can then reinterpret the truncation as an ansatz, keeping both supersymmetries, that allows us to solve the BPS equations of the full $N = 2$ theory.* In this way we find D-term solutions of $N = 1$ with FI terms, embedded in $N = 2$ supergravity.

Equation (2.19) can be used to obtain a moment map with these properties, which we illustrate by means of examples involving the two normal quaternionic-Kähler manifolds of (quaternionic) dimension one. This can be generalized to normal quaternionic manifolds of higher dimension as they always contain as a completely geodesic submanifold one of the two quaternionic manifolds of dimension one equipped with an induced quaternionic structure [36 – 38].

3.2 Example of $\frac{\mathrm{Sp}(1,1)}{\mathrm{Sp}(1)\mathrm{Sp}(1)}$

Details on the geometry and coset parametrization of the coset space $\frac{\mathrm{Sp}(1,1)}{\mathrm{Sp}(1)\mathrm{Sp}(1)}$ are given in appendix B. The space is characterized by the following metric:

$$
\mathrm{d}s^2 = (\mathrm{d}h)^2 + \mathrm{e}^{-2h}\left[(\mathrm{d}b^1)^2 + (\mathrm{d}b^2)^2 + (\mathrm{d}b^3)^2\right] . \tag{3.1}
$$

Consider now the Killing vector[5]

$$k_1 = 2b^1 \frac{\partial}{\partial b^2} - 2b^2 \frac{\partial}{\partial b^1}. \tag{3.2}$$

It rotates the SU(2) connection as follows:

$$\mathcal{L}_{k_1} \omega^x = -2\varepsilon^{xyz} \omega^y \delta_3^z = -\nabla \delta_3^x, \tag{3.3}$$

which implies that the compensator r_{k_1} is a constant: $r_{k_1}^x = 2\delta_3^x$. The moment map can be computed as

$$\mathcal{P}_{k_1} = \iota_{k_1} \omega + \frac{1}{2} r_{k_1} = \mathrm{e}^{-h} \begin{pmatrix} b^2 \\ -b^1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.4}$$

To have a vanishing superpotential, we impose the condition $\mathcal{P}^1 = \mathcal{P}^2 = 0$:

$$b^1 = b^2 = 0. \tag{3.5}$$

This configuration defines a consistent truncation to a Kähler-Hodge submanifold of $\frac{\mathrm{Sp}(1,1)}{\mathrm{Sp}(1)\mathrm{Sp}(1)}$:

$$\frac{\mathrm{Sp}(1,1)}{\mathrm{Sp}(1)\mathrm{Sp}(1)} \xrightarrow{b^1 = b^2 = 0} \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}. \tag{3.6}$$

One can check indeed that three of the ten isometries of the quaternionic space are preserved by the truncation (the shift of b^3, the dilatation coming from the Cartan generator, and one of the compact generators; the latter is the generator that we will gauge), and that they form the algebra $\mathrm{SU}(1,1)$. $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$ can be parametrized by the complex field

$$\Phi = -b^3 + \mathrm{i}\mathrm{e}^h, \tag{3.7}$$

in terms of Φ the metric can be written as

$$\mathrm{d}s^2 = \frac{\mathrm{d}\Phi \mathrm{d}\overline{\Phi}}{(\mathrm{Im}\,\Phi)^2}. \tag{3.8}$$

If we now use as Killing vector ξk_1, with ξ an arbitrary real constant, the $N = 1$ potential will have a constant Fayet-Iliopoulos term given by the D-term

$$D = 2g\mathcal{P}^3 = 2g\xi. \tag{3.9}$$

We denote this quantity by D as it is the D-term of $N = 1$ (the normalization will be explained in section 6.1). In this way, an *arbitrary* FI constant can be introduced in the D-term potential of the reduced $N = 1$ theory. In this reduced theory, k_1 identically vanishes so that it does not act on $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$. Therefore, the only effect of the gauging is the generation of an FI term.

[5]The Killing vector (3.2) is one that rotates the quaternionic structure, while its invariant subspace will define the truncated manifold. Furthermore, it preserves the J^3 complex structure, which is the one that will play the role of complex structure in the truncated, $N = 1$, theory.

To generate a D-term potential with the right properties for D-term string solutions, we gauge a linear combination of k_1 and the following one, which is the uplift of the compact isometry of $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$:

$$k_2 = 4b^3\frac{\partial}{\partial h} + 4b^1b^3\frac{\partial}{\partial b^1} + 4b^2b^3\frac{\partial}{\partial b^2} + 2\left[(b^3)^2 - \mathrm{e}^{2h} + 1 - (b^1)^2 - (b^2)^2\right]\frac{\partial}{\partial b^3}, \tag{3.10}$$

with corresponding moment map

$$\mathcal{P}_{k_2} = \begin{pmatrix} -2b^2 - 2b^1b^3\mathrm{e}^{-h} \\ 2b^1 - 2b^2b^3\mathrm{e}^{-h} \\ -\mathrm{e}^{-h}\left[(b^3)^2 + 1 - (b^2)^2 - (b^1)^2\right] - \mathrm{e}^h \end{pmatrix}. \tag{3.11}$$

These Killing vectors automatically satisfy the requirements of [27, 28].

Gauging the linear combination $k = k_2 + \xi k_1$ and imposing the truncation (3.5) then results in an $N = 1$ theory with vanishing superpotential and D-term

$$D = -2g\left[\mathrm{e}^{-h}(b^3)^2 + \mathrm{e}^{-h} + \mathrm{e}^h\right] + 2g\xi. \tag{3.12}$$

3.3 Example of $\frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\,\mathrm{U}(1)}$

The metric can be found using the solvable algebra approach (see appendix B) or in the literature, for example in the work of [39]. It is given by:

$$\mathrm{d}s^2 = \frac{1}{2}\mathrm{d}h^2 + \frac{1}{2}\mathrm{e}^{-2h}\left(\mathrm{d}b^3 - e^1\mathrm{d}e^2 + e^2\mathrm{d}e^1\right)^2 + \mathrm{e}^{-h}\left[(\mathrm{d}e^1)^2 + (\mathrm{d}e^2)^2\right]. \tag{3.13}$$

Killing vectors and moment maps can be found in [39]. The correspondence with our notation is given in (B.36). Setting

$$e^1 = e^2 = 0, \tag{3.14}$$

the space can be consistently "truncated" to the submanifold $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$ in the upper half-plane parametrization. We define the complex field Φ by

$$\Phi = b^3 + \mathrm{i}\mathrm{e}^h, \tag{3.15}$$

leading to the metric differing from (3.8) by a normalization factor:

$$\mathrm{d}s^2 = \frac{\mathrm{d}\Phi\mathrm{d}\overline{\Phi}}{2(\mathrm{Im}\,\Phi)^2}. \tag{3.16}$$

This different normalization is due to another embedding of the R-symmetry SU(2) in the two quaternionic-Kähler manifolds.

Along the lines of the previous section, we gauge a combination

$$k = 2\xi k_4 + 2k_6 + 2k_1, \tag{3.17}$$

following the labelling in [39]. The Killing vector in the basis $(h,\, b^3,\, e^1,\, e^2)$ is

$$k = 2\xi \begin{pmatrix} 0 \\ 0 \\ e^2 \\ -e^1 \end{pmatrix} - 2 \begin{pmatrix} 2b^3 \\ 1+(b^3)^2 - \left(\mathrm{e}^h + \frac{1}{2}E\right)^2 \\ -b^3 e^1 + e^2 \left(\mathrm{e}^h + \frac{1}{2}E\right) \\ b^3 e^2 - e^1 \left(\mathrm{e}^h + \frac{1}{2}E\right) \end{pmatrix}, \tag{3.18}$$

where $E \equiv (e^1)^2 + (e^2)^2$. The moment map is then

$$\mathcal{P} = \xi \begin{pmatrix} -\sqrt{2}\mathrm{e}^{-h/2} e^2 \\ \sqrt{2}\mathrm{e}^{-h/2} e^1 \\ 1 - \frac{1}{2}\mathrm{e}^{-h} E \end{pmatrix} + \begin{pmatrix} -\sqrt{2}\mathrm{e}^{-h/2}\left[b^3 e^1 + e^2\left(-\mathrm{e}^h + \frac{1}{2}E\right)\right] \\ \sqrt{2}\mathrm{e}^{-h/2}\left[-b^3 e^2 + e^1\left(-\mathrm{e}^h + \frac{1}{2}E\right)\right] \\ -\frac{1}{2}\mathrm{e}^h - \frac{1}{2}\mathrm{e}^{-h}\left[1+(b^3)^2 + \frac{1}{4}E^2\right] + \frac{3}{2}E \end{pmatrix} \tag{3.19}$$

After truncation (3.14), we get the following D-term

$$D = -g\left[(b^3)^2\mathrm{e}^{-h} + \mathrm{e}^{-h} + \mathrm{e}^h\right] + 2g\xi. \tag{3.20}$$

3.4 Common formulae in the truncated space

The data of the two models lead to common formulae in the truncated space. The quaternionic vielbein as obtained from appendix B, and with the conditions (3.5) or (3.14), reduces to

$$f^{11} = -\mathrm{i}\alpha\frac{\mathrm{d}\Phi}{2\,\mathrm{Im}\,\Phi}, \qquad f^{22} = \mathrm{i}\alpha\frac{\mathrm{d}\bar\Phi}{2\,\mathrm{Im}\,\Phi}, \qquad f^{12} = f^{21} = 0, \tag{3.21}$$

where

$$\alpha = \sqrt{2} \quad \text{for } \frac{\mathrm{Sp}(1,1)}{\mathrm{Sp}(1)\mathrm{Sp}(1)}, \qquad \alpha = 1 \quad \text{for } \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\,\mathrm{U}(1)}. \tag{3.22}$$

In terms of the complex field Φ, the metric is

$$\mathrm{d}s^2 = g_{XY}\mathrm{d}q^X\mathrm{d}q^Y = \frac{\alpha^2}{2(\mathrm{Im}\,\Phi)^2}\mathrm{d}\Phi\mathrm{d}\bar\Phi. \tag{3.23}$$

The Killing vector k reduces to

$$k = -2(\Phi^2+1)\partial_\Phi + \text{c.c.} \tag{3.24}$$

The D-term in terms of Φ is given by

$$D \equiv 2g\mathcal{P}^3 = -g\alpha^2\frac{|\Phi|^2+1}{\mathrm{Im}\,\Phi} + 2g\xi. \tag{3.25}$$

Note that both our examples give the same D-term, up to a normalization in the first term. The normalization issue is not related to remaining fields in this reduction, but is due to the non-Abelian aspects of the R-symmetry SU(2) in the different quaternionic-Kähler manifolds, leading to a different normalization for

$$\omega^3 = \alpha^2\frac{\mathrm{d}\Phi + \mathrm{d}\bar\Phi}{8\,\mathrm{Im}\,\Phi}. \tag{3.26}$$

The complex field Φ belongs to a chiral multiplet in the resulting $N=1$ theory and parametrizes the *non-trivial* Kähler space $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$. Further on, we will show that the $N=2$ BPS equations reduce correctly to those of this $N=1$ system.

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4. The $N = 2$ BPS equations

We study the $N = 2$ BPS equations for a system with one vector- and one hypermultiplet, and the truncation to $N = 1$. The hypermultiplet target space and gauging were described above. Below, we define the special geometry of the vector multiplet target space.

4.1 Choice of special geometry

The special geometry that we consider is the *minimal* one defined by the quadratic prepotential

$$F(X^0, X^1) = -\frac{1}{2}\mathrm{i}\left[X^0X^0 - X^1X^1\right]. \tag{4.1}$$

This leads to $N_{IJ} = 2\eta_{IJ}$ with $\eta_{IJ} = \mathrm{diag}(-1, 1)$. Introducing the special coordinate $z = \frac{X^1}{X^0}$, we obtain the Kähler potential (2.8)

$$\mathcal{K} = -\log\left[2(1 - z\bar{z})\right], \qquad g_{z\bar{z}} = (1 - z\bar{z})^{-2}, \tag{4.2}$$

where we replace the only value of the index α with z. This corresponds to the coset $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$. We also obtain the vector kinetic matrix

$$\mathcal{N}_{IJ} = -\frac{\mathrm{i}}{1 - z^2}\begin{pmatrix} 1 + z^2 & -2z \\ -2z & 1 + z^2 \end{pmatrix}. \tag{4.3}$$

At the base point $z = 0$, which we will use in the reduction to $N = 1$, we have thus $\mathrm{Im}\,\mathcal{N}_{IJ} = -\delta_{IJ}$ and $\mathrm{Re}\,\mathcal{N}_{IJ} = 0$.

The symplectic section obtained from the prepotential according to (2.3) is

$$V = \begin{pmatrix} X^0 \\ X^1 \\ -\mathrm{i}X^0 \\ \mathrm{i}X^1 \end{pmatrix} = \mathrm{e}^{\mathcal{K}/2}\begin{pmatrix} 1 \\ z \\ -\mathrm{i} \\ \mathrm{i}z \end{pmatrix}, \qquad \mathcal{D}_z V = \frac{\mathrm{e}^{\mathcal{K}/2}}{1 - z\bar{z}}\begin{pmatrix} \bar{z} \\ 1 \\ -\mathrm{i}\bar{z} \\ \mathrm{i} \end{pmatrix}. \tag{4.4}$$

We intend to use the vector $W_\mu \equiv W^1_\mu$ to gauge the appropriate isometry with the Killing vector k. To compute the scalar potential we need the component U^{11} in (2.29) which is easily computed using (4.4) such that

$$U^{11} - 3X^1\bar{X}^1 = \mathrm{e}^{\mathcal{K}}(1 - 3z\bar{z}) = \frac{1 - 3z\bar{z}}{2(1 - z\bar{z})}. \tag{4.5}$$

The scalar potential is then given by

$$g^{-2}\mathcal{V} = \frac{2(1 - 3z\bar{z})}{1 - z\bar{z}}\mathcal{P}^x\mathcal{P}^x + \frac{z\bar{z}}{(1 - z\bar{z})}g_{XY}k^Xk^Y, \tag{4.6}$$

where $\mathcal{P}^x$ is the moment map that corresponds to the Killing vector k.

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4.2 Ansatz for the bosonic fields in $N = 2$

Motivated by the fact that we want our $N = 2$ bosonic action to reduce to the one of an $N = 1$ theory, we look for a field configuration that effectively truncates the $N = 2$ action with vector- and hypermultiplet to an $N = 1$ action with vector (i.e. gauge) and chiral multiplet. Consistent truncations of $N = 2$ to $N = 1$ are studied in [27], to which we refer for details. The consistency conditions derived there come from demanding:

$$\delta\psi_{2\mu} = 0 \quad \text{with} \quad \epsilon_2 = 0, \tag{4.7}$$

(similarly for the other truncated fermions). This choice is consistent with the survival of the complex structure J^3 in the reduced theory. Furthermore, we demand that the sources of truncated bosonic fields vanish. The conditions can be satisfied by imposing the following conditions on the bosonic field configuration:

1. The scalar z of the vector multiplet vanishes on the configuration.
2. We gauge a U(1) isometry of the quaternionic-Kähler manifold with the vector field $W \equiv W^1$. The gauging is Abelian (therefore the symmetries of the special manifold are not gauged) and the bare graviphoton does not gauge any symmetries and is put to zero: $W^0 = 0$.
3. The only non-vanishing components of the moment map $\mathcal{P}^x$ and the quaternionic SU(2) connections ω^x are $\mathcal{P}^3$ and ω^3, respectively.

These conditions are implemented as follows:

$$\begin{aligned} z = W^0 &= 0, \\ k_0^X = k_I^z &= 0, \\ \mathcal{P}^1 = \mathcal{P}^2 &= 0, \\ \omega^1 = \omega^2 &= 0. \end{aligned} \tag{4.8}$$

Note that $z = 0$ is a critical point of the scalar potential (4.6).

With our choice of special geometry based on the quadratic prepotential (4.1), the conditions above imply that on the configuration

$$g_{z\bar{z}} = 1, \quad \mathcal{D}_z X^I = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathcal{N}_{IJ} = -\mathrm{i} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.9}$$

This implies:

$$S^{ij} = \mathcal{N}^{iA} = T^-_{\mu\nu} = A_\mu = 0. \tag{4.10}$$

The non-vanishing data are

$$V_i^{\ j} = \mathrm{i}\left(\omega^3 + gW\mathcal{P}^3\right)\left(\sigma^3\right)_i^{\ j}, \qquad N_{ij}^z = \frac{1}{\sqrt{2}}\mathrm{i}\mathcal{P}^3\left(\sigma^3\right)_{ij}, \tag{4.11}$$

where

$$(\sigma^3)_i{}^j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\sigma^3)_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.12}$$

The data of the quaternionic manifold are in section 3.4.

On the configuration defined above, the bosonic part of the $N = 2$ action can be obtained from (2.1) minus the potential of (4.6). This gives

$$e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\alpha^2}{4(\operatorname{Im}\Phi)^2}\nabla_\mu\Phi\nabla^\mu\bar{\Phi} - 2g^2\left[\alpha^2\frac{|\Phi|^2+1}{2\operatorname{Im}\Phi} - \xi\right]^2, \tag{4.13}$$

where

$$\nabla_\mu\Phi = \partial_\mu\Phi - 2gW_\mu\left(\Phi^2+1\right). \tag{4.14}$$

Note that since the ansatz satisfies the conditions for a consistent truncation, solutions of the field equations derived from this action are solutions of the full $N = 2$ field equations. This is due to the fact that the truncated fields appear at least quadratically in the $N = 2$ action.

The supersymmetry transformations (2.22) become, using (4.8) and (4.10),

$$\begin{aligned}
\delta\psi^i_\mu &= \nabla_\mu(\omega)\epsilon^i = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\epsilon^i + V_{\mu j}{}^i\epsilon^j, \\
\delta\lambda^z_i &= -\frac{1}{2}g^{z\bar{z}}\mathcal{D}_{\bar{z}}\bar{X}^1 \operatorname{Im}\mathcal{N}_{11}F^{-1}_{\mu\nu}\gamma^{\mu\nu}\varepsilon_{ij}\epsilon^j + gN^z_{ij}\epsilon^j, \\
\delta\zeta^A &= \frac{1}{2}\mathrm{i}f^{Ai}_X\not{\nabla}q^X\epsilon_i.
\end{aligned} \tag{4.15}$$

Using (4.7), (4.9) and (4.11), this gives us

$$\begin{aligned}
\delta\psi^1_\mu &= (\partial_\mu + \frac{1}{4}\omega_{\mu|ab}\gamma^{ab} + \frac{1}{2}\mathrm{i}A^B_\mu)\,\epsilon^1, & \delta\psi^2_\mu &= (\partial_\mu + \frac{1}{4}\omega_{\mu|ab}\gamma^{ab} - \frac{1}{2}\mathrm{i}A^B_\mu)\,\epsilon^2, \\
\delta\lambda_2 &= -\frac{1}{2\sqrt{2}}F^-_{\mu\nu}\gamma^{\mu\nu}\epsilon^1 - \mathrm{i}\frac{1}{\sqrt{2}}D\,\epsilon^1, & \delta\lambda_1 &= \frac{1}{2\sqrt{2}}F^-_{\mu\nu}\gamma^{\mu\nu}\epsilon^2 - \mathrm{i}\frac{1}{\sqrt{2}}D\,\epsilon^2, \\
\delta\zeta^1 &= \frac{\alpha}{4\operatorname{Im}\Phi}\not{\nabla}\Phi\epsilon_1, & \delta\zeta^2 &= -\frac{\alpha}{4\operatorname{Im}\Phi}\not{\nabla}\bar{\Phi}\epsilon_2.
\end{aligned} \tag{4.16}$$

In these equations A^B_μ is the matter connection of the gravitini on the configuration:[6]

$$A^B_\mu = 2\omega^3_\mu + W_\mu D = \frac{\alpha^2\left(\partial_\mu\Phi + \partial_\mu\bar{\Phi}\right)}{4\operatorname{Im}\Phi} + W_\mu D. \tag{4.17}$$

We see that the equations for ϵ_1 and ϵ_2 split into two sets. Starting from the $N = 2$ action, we have defined a consistent truncation $N = 2 \to N = 1$: once we take $\epsilon_2 \equiv 0$, the BPS equations for ϵ_1, $\delta\psi^1_\mu = \delta\lambda_2 = \delta\zeta^1 = 0$, correspond to the BPS equations of an $N = 1$ supergravity theory with vanishing superpotential, a D-term, a constant effective coupling for the vector field kinetic term, and a U(1) gauging of the isometry $\delta\Phi = 2g(\Phi^2+1)$ of the upper half plane. We verify this explicitly in section 6.1.

[6]In general for such reductions to $N = 1$ using only ϵ^1, we have $A^B_\mu = A_\mu - 2\mathrm{i}V_{\mu 1}{}^1$.

However, if the equations for ϵ_2, $\delta\psi_\mu^2 = \delta\lambda_1 = \delta\zeta^2 = 0$, can be simultaneously solved with those for ϵ_1, we have an $N = 2$ BPS solution with an extra supersymmetry corresponding to ϵ_2. In the next section we will show this to be the case for a (1/2)-projection of both supersymmetries. From now on we will continue with both sets of BPS equations (for ϵ_1 and ϵ_2) of the full N=2 theory.

For similar results on such an extra supersymmetry arising from an embedding of $N = 1$ into $N = 2$ in the context of global supersymmetry, see [40] and [41].

5. Finding a BPS cosmic string solution

We now proceed to solve the resulting BPS equations for a cosmic string ansatz. We study the BPS equations for ϵ_1 and ϵ_2 together.

5.1 String ansatz; projector and integrability conditions

We assume a straight, static cosmic string on the z-axis. We use cylindrical coordinates (t, z, r, θ). We take the following ansatz for the spacetime metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + C^2(r)\mathrm{d}\theta^2. \tag{5.1}$$

The vielbeins are (we take $C(r) > 0$ without loss of generality)

$$\hat{e}^1 = \mathrm{d}r, \qquad \hat{e}^2 = C(r)\mathrm{d}\theta, \tag{5.2}$$

from which we can deduce the spin connection

$$\omega_r^{12} = 0, \qquad \omega_\theta^{12} = -C'(r) \equiv -\frac{\mathrm{d}\,C(r)}{\mathrm{d}\,r}. \tag{5.3}$$

The complex field is independent of z: $\Phi = \Phi(r, \theta)$.

Squaring the BPS equations for the chiral fermions, one gets a consistency condition for the projectors on the Killing spinors. The projector condition can also be derived from the integrability conditions

$$(C''\gamma^{12} - \mathrm{i}F^B_{r\theta})\epsilon^1 = 0, \qquad (C''\gamma^{12} + \mathrm{i}F^B_{r\theta})\epsilon^2 = 0. \tag{5.4}$$

One obtains in this way that

$$\gamma^{12}\epsilon^1 = \mp\mathrm{i}\epsilon^1, \qquad \gamma^{12}\epsilon^2 = \pm\mathrm{i}\epsilon^2. \tag{5.5}$$

(so ϵ^1 and ϵ^2 have opposite chirality on the string worldsheet). It follows that the string configuration preserves a maximum of 4 supercharges out of the 8 supercharges of the $N = 2$ supergravity system. Imposing this projector condition gives the following BPS equations, which follow from (4.16):

$$\left(\partial_\mu \mp \frac{\mathrm{i}}{2}\omega_{\mu|12} + \frac{\mathrm{i}}{2}A^B_\mu\right)\epsilon^1 = 0, \tag{5.6}$$

$$\left(\partial_\mu \pm \frac{\mathrm{i}}{2}\omega_{\mu|12} - \frac{\mathrm{i}}{2}A^B_\mu\right)\epsilon^2 = 0, \tag{5.7}$$

$$\mp C^{-1}F_{r\theta} + D = 0, \tag{5.8}$$

$$(\nabla_r \pm \mathrm{i}C^{-1}\nabla_\theta)\Phi = 0, \tag{5.9}$$

and the integrability condition

$$C'' \pm F^B_{r\theta} = 0. \tag{5.10}$$

5.2 Solving the BPS equations

To find a half BPS cosmic string solution of the $N = 2$ theory, we attempt to solve the BPS equations given above.

Hyperini BPS equations and ansatz for the scalar

To solve the hyperini BPS equation (5.9), we will follow [42] by defining a holomorphic derivative on the plane perpendicular to the cosmic string. This is possible because any two dimensional metric is Kähler and therefore admits a complex structure. This property has been used to obtain BPS equations for cosmic strings by Comtet and Gibbons [43], see also Ruback [44]. The use of holomorphic derivatives will give us a nice way to get the right ansatz for the scalar field. The method will be seen to amount to a coordinate transformation of the upper half plane to the unit disk. Let us define

$$z = \exp\left[\int \frac{\mathrm{d}r}{C(r)} + \mathrm{i}\theta\right]. \tag{5.11}$$

With these coordinates the 2-dimensional metric is $\mathrm{d}s^2 = \Omega^2 \mathrm{d}z\mathrm{d}\bar{z}$, where Ω is the conformal factor that reduces to 1 when $z = 0$. We then have

$$z\partial_z = C\partial_r \mp \mathrm{i}\partial_\theta, \qquad \bar{z}\partial_{\bar{z}} = C\partial_r \pm \mathrm{i}\partial_\theta, \tag{5.12}$$

and

$$zW_z = CW_r \mp \mathrm{i}W_\theta, \qquad \bar{z}W_{\bar{z}} = CW_r \pm \mathrm{i}W_\theta. \tag{5.13}$$

We can then write the hyperini BPS equation (5.9) as

$$\nabla_{\bar{z}}\Phi = \partial_{\bar{z}}\Phi + gk^\Phi W_{\bar{z}} = 0, \qquad k^\Phi = -2\left(\Phi^2 + 1\right). \tag{5.14}$$

where $\delta\Phi = -gk^\Phi$, in our case $\delta\Phi = 2g(\Phi^2 + 1)$. We can now solve the equation for W_z:

$$2gW_{\bar{z}} = \frac{\partial_{\bar{z}}\Phi}{\Phi^2 + 1} = \partial_{\bar{z}} \tan^{-1}(\Phi) \tag{5.15}$$

Using the identity

$$\tan^{-1}\Phi = \frac{\mathrm{i}}{2}\log\frac{\mathrm{i} + \Phi}{\mathrm{i} - \Phi}, \tag{5.16}$$

we have

$$2gW_{\bar{z}} = -\frac{\mathrm{i}}{2}\partial_{\bar{z}} \log u, \tag{5.17}$$

where we define

$$u = \frac{\mathrm{i} - \Phi}{\mathrm{i} + \Phi}, \qquad \text{that is} \qquad \Phi = \mathrm{i}\frac{1 - u}{1 + u}. \tag{5.18}$$

The expression (5.17) is familiar from the study of Abelian vortices in flat space. The variable u is convenient to analyse the BPS equations, as we see by looking at the gauge transformation and the D-term. In the u-plane, the gauge transformation is a change of phase with charge $+4g$:

$$\delta u = 4gi u, \tag{5.19}$$

and the D-term is a function of $|u|^2$:

$$D = -2g\alpha^2 \frac{1+|u|^2}{1-|u|^2} + 2g\xi. \tag{5.20}$$

If $\xi > \alpha^2$, the moduli of vacua in the u-plane is a circle centered at the origin ($u = 0$) with its radius fixed by the FI term ξ:

$$D = 0 \iff |u|^2 = \frac{\xi - \alpha^2}{\xi + \alpha^2}. \tag{5.21}$$

Note that, if $\xi = \alpha^2$, there is a unique vacuum $u = 0$ and the U(1) gauge symmetry is not spontaneously broken. For $\xi < \alpha^2$ the vacuum is an unstable de Sitter solution. In what follows we assume $\xi > \alpha^2$.

For a cosmic string located at the origin, we now take the following ansatz:

$$u = f(r)\mathrm{e}^{\mathrm{i}n\theta}, \quad f(0) = 0, \quad f(\infty) \to \sqrt{\frac{\xi - \alpha^2}{\xi + \alpha^2}}. \tag{5.22}$$

Using the identity

$$\log u = \frac{1}{2}\log|u|^2 + \mathrm{i}(\arg u + 2\pi m), \tag{5.23}$$

it follows that

$$2gW_{\bar z} = -\frac{1}{2}\mathrm{i}\partial_{\bar z}\log u = -\frac{1}{2}\mathrm{i}\left(\frac{1}{2}C(r)\partial_r \log|u|^2 \mp C^{-1}n\right), \tag{5.24}$$

or in other words

$$W_r = 0, \quad \mp 2gW_\theta = \frac{1}{2}\left[C(r)\frac{f'(r)}{f(r)} \mp n\right]. \tag{5.25}$$

Gaugini BPS equation

We can now solve the BPS equations of the gaugini. First we need to compute the field strength. As we work in the gauge $W_r = 0$, we have

$$F_{r\theta} = \partial_r W_\theta, \tag{5.26}$$

and the gaugini BPS equation is

$$\pm W_\theta'(r) = C(r)\, D. \tag{5.27}$$

Gravitini BPS equations

The gravitini BPS equations are

$$\left(\partial_r + \frac{\mathrm{i}}{2}A^B_r\right)\epsilon^1 = 0, \qquad \left[\partial_\theta \pm \frac{\mathrm{i}}{2}C'(r) + \frac{\mathrm{i}}{2}A^B_\theta\right]\epsilon^1 = 0, \tag{5.28}$$

$$\left(\partial_r - \frac{\mathrm{i}}{2}A^B_r\right)\epsilon^2 = 0, \qquad \left[\partial_\theta \mp \frac{\mathrm{i}}{2}C'(r) - \frac{\mathrm{i}}{2}A^B_\theta\right]\epsilon^2 = 0\,, \tag{5.29}$$

with the integrability condition

$$C'' \pm F^B_{r\theta} = 0. \tag{5.30}$$

These equations are different from those solved in [1], because here the radial component A^B_r of the gravitini connection A^B_μ does not vanish. This can be traced back to the non-vanishing of the radial component of the Kähler connection of the half-plane. The complex scalar parametrizes the Kähler-Hodge manifold $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$ with the upper half plane Kähler potential:

$$\mathcal{K}^{\mathrm{HP}} = -\alpha^2 \log -\mathrm{i}(\Phi - \overline{\Phi}). \tag{5.31}$$

Under the change of variables (5.18) it gets the form

$$\mathcal{K}^{\mathrm{HP}} = -\alpha^2 \log \frac{2(1-u\bar{u})}{(1+u)(1+\bar{u})} = -\alpha^2 \log 2(1-u\bar{u}) + \alpha^2\log(1+u) + \alpha^2 \log(1+\bar{u}). \tag{5.32}$$

The Kähler potential $\mathcal{K}^{\mathrm{HP}}$ is not invariant under the symmetry $\delta\Phi = 2g\left(\Phi^2+1\right)$, but the manifold $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$ admits another Kähler description in term of the unit disk Kähler potential

$$\mathcal{K}^{\mathrm{UD}} = -\alpha^2 \log 2(1-u\bar{u}), \tag{5.33}$$

which is invariant under the symmetry $\delta u = 4giu$. As we have

$$\mathcal{K}^{\mathrm{HP}} = \mathcal{K}^{\mathrm{UD}} + \ell + \bar{\ell} \quad \text{with} \quad \ell(u) = \alpha^2\log(1+u), \tag{5.34}$$

we see that the upper half plane and the unit disk are related by a Kähler transformation generated by the analytic function $\ell(u)$. As is well known, the Kähler metric does not change under a Kähler transformation, however it is the Kähler U(1)-connection that enters the BPS equation of the gravitino

$$\mathcal{Q} = -\frac{\mathrm{i}}{2}\left(\mathrm{d}\phi\frac{\partial\mathcal{K}}{\partial\phi} - \mathrm{d}\overline{\phi}\frac{\partial\mathcal{K}}{\partial\overline{\phi}}\right), \qquad A^B = \mathcal{Q} + WD, \tag{5.35}$$

and the latter transforms as

$$\mathcal{Q} \to \mathcal{Q} - \frac{\mathrm{i}}{2}\left(\mathrm{d}\phi\frac{\partial\ell}{\partial\phi} - \mathrm{d}\bar{\phi}\frac{\partial\bar{\ell}}{\partial\overline{\phi}}\right) = \mathcal{Q} + \mathrm{d}\,\mathrm{Im}\,\ell. \tag{5.36}$$

From the point of view of $N=2$ supergravity, the Kähler potential comes from the metric of the quaternionic-Kähler manifold, and the choice of the half-plane was imposed by the value of the quaternionic SU(2)-connection ω^x which becomes the U(1)-connection of the Kähler-Hodge manifold defined by the string configuration.

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Clearly, the Kähler transformation of the Kähler-Hodge manifold is related to a change of gauge of the SU(2)-connection ω^x, which is in the normalization of [45]

$$\omega^x \to \omega^x - \frac{1}{2}\nabla \ell^x. \tag{5.37}$$

As we have $\omega^1 = \omega^2 = 0$ on the string configuration, we see that if $r^1 = r^2 = 0$ on the string configuration, the SU(2)-redefinition of the quaternionic connection is seen as a Kähler transformation for the Kähler-Hodge manifold defined by the string configuration. Comparing with (4.17), we have

$$\mathrm{Im}\,\ell = -\ell^3. \tag{5.38}$$

Using the analytic property of ℓ, the transformation of the U(1) connection under the Kähler transformation generated by ℓ can be rewritten as:

$$\mathcal{Q} \to \mathcal{Q} - \mathrm{d}\left(\mathrm{i}\frac{\ell - \bar{\ell}}{2}\right), \tag{5.39}$$

so that we have

$$\mathcal{Q}^{\mathrm{HP}} = \mathcal{Q}^{\mathrm{UD}} - \mathrm{id}\left(\frac{\ell - \bar{\ell}}{2}\right), \tag{5.40}$$

where $\mathcal{Q}^{\mathrm{UD}}$ is the U(1)-connection of the unit disk:

$$\mathcal{Q}^{\mathrm{UD}} = \mathrm{i}\frac{\alpha^2}{2}\frac{u\mathrm{d}\overline{u} - \mathrm{d}u\overline{u}}{1 - u\overline{u}}. \tag{5.41}$$

To solve the BPS equation we use $u = f(r)\mathrm{e}^{\mathrm{i}n\theta}$. This implies for the unit disk:

$$\mathcal{Q}^{\mathrm{UD}}_r = 0, \qquad \mathcal{Q}^{\mathrm{UD}}_\theta = n\alpha^2\frac{f^2}{1-f^2}, \tag{5.42}$$

whereas the half-plane gets an extra contribution coming from ℓ, which introduces a dependence on the azimuthal angle. As $W_r = 0$, A^B is just a function of the radius r in the case of the unit disk and the gravitini BPS equations give differential equations for the profile functions $f(r)$ and $C(r)$ depending only on one variable r.

The situation is not that nice for the half-plane, due to the presence of ℓ, see (5.34), which depends explicitly on the azimuthal angle θ. It is interesting to note that we could actually work with the unit disk Kähler potential if we had defined our quaternionic structure with the SU(2)-connection $\omega^x + \frac{1}{2}\mathrm{d}(\mathrm{Im}\,\ell)$.

To solve the BPS equations for the half-plane, we redefine the Killing spinors by a rotation, in order to make the gravitini BPS equations independent of the azimuthal angle:

$$\epsilon^1 = \exp\left(\frac{\mathrm{i}}{2}\,\mathrm{Im}\ \ell\right)\tilde{\epsilon}^1, \qquad \epsilon^2 = \exp\left(-\frac{\mathrm{i}}{2}\,\mathrm{Im}\ \ell\right)\tilde{\epsilon}^2 \tag{5.43}$$

Such a redefinition has the same effect as a Kähler transformation. Indeed, the gravitini BPS equations for $\tilde{\epsilon^i}$ are

$$\left[\mathcal{D} + \frac{1}{2}\mathrm{id}(\mathrm{Im}\ \ell) + \frac{1}{2}\mathrm{i}A^B\right]\tilde{\epsilon}^1 = 0, \qquad \left[\mathcal{D} - \frac{1}{2}\mathrm{id}(\mathrm{Im}\ \ell) - \frac{1}{2}\mathrm{i}A^B\right]\tilde{\epsilon}^2 = 0, \tag{5.44}$$

where $\mathcal{D}$ is a derivative including spin connection. Defining

$$\tilde{A}^B = A^B + \mathrm{d}(\mathrm{Im}\ \ell), \tag{5.45}$$

we have

$$\tilde{A}^B = \mathcal{Q}^{\mathrm{UD}} + W\, D. \tag{5.46}$$

In terms of $\tilde{\epsilon}^i$ and $\tilde{A}^B$, the gravitini BPS equations have the same structure as in [1]:

$$\partial_r \tilde{\epsilon}^1 = 0 \qquad \left[\partial_\theta \pm \frac{1}{2}\mathrm{i}C'(r) + \frac{1}{2}\mathrm{i}\tilde{A}^B_\theta\right]\tilde{\epsilon}^1 = 0, \tag{5.47}$$

$$\partial_r \tilde{\epsilon}^2 = 0 \qquad \left[\partial_\theta \mp \frac{1}{2}\mathrm{i}C'(r) - \frac{1}{2}\mathrm{i}\tilde{A}^B_\theta\right]\tilde{\epsilon}^2 = 0, \tag{5.48}$$

where $\tilde{A}^B_\theta$ depends only on the radial distance r.

We can now follow the treatment of [1] to solve the BPS equations. Globally well defined spinors are

$$\tilde{\epsilon}^1 = \mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta}\tilde{\epsilon}^1_0 \iff \tilde{\epsilon}_2 = \mathrm{e}^{\pm\frac{\mathrm{i}}{2}\theta}\tilde{\epsilon}^2_0, \tag{5.49}$$

where ϵ^i_0 is a constant spinor that satisfies the same projection relation as ϵ^i.

Therefore, the gravitini BPS equations are equivalent to the differential equation

$$C' = 1 \mp \tilde{A}^B_\theta. \tag{5.50}$$

and the Killing spinors ϵ^i are related to $\tilde{\epsilon}^i$ by a non-constant shift of phase:

$$\epsilon^1 = \mathrm{e}^{\mathrm{i}\Delta(r,\theta)}\ \tilde{\epsilon}^1, \quad \epsilon^2 = \mathrm{e}^{-\mathrm{i}\Delta(r,\theta)}\ \tilde{\epsilon}^2, \tag{5.51}$$

where

$$\Delta(r,\theta) = \frac{1}{2}\mathrm{Im}\ \ell = -\frac{\mathrm{i}\alpha^2}{4}\log\frac{1+u}{1+\overline{u}} \tag{5.52}$$

$$= \frac{\alpha^2}{2}\arg(1+u) = \frac{\alpha^2}{2}\tan^{-1}\left(\frac{f(r)\sin n\theta}{1+f(r)\cos n\theta}\right). \tag{5.53}$$

When $r\to\infty$, $\Delta(r,\theta)$ does not depend on r anymore, as $f \to \sqrt{\frac{\xi-\alpha^2}{\xi+\alpha^2}}$.

5.3 Profile of the string

From equations (5.25), (5.27) and (5.50) we obtain the equations that determine the profile of the string:

$$\begin{aligned}\pm f'(r) &= \frac{f(r)}{C(r)}\left(n - 4gW_\theta(r)\right),\\ \pm W'_\theta(r) &= C(r)D(r),\\ C'(r) &= 1 \mp \tilde{A}^B_\theta(r),\end{aligned} \tag{5.54}$$

with

$$\tilde{A}^B_\theta = \mathcal{Q}^{\mathrm{UD}}_\theta + W_\theta D,$$

$$
\begin{aligned}
\mathcal{Q}_\theta^{\mathrm{UD}} &= n\alpha^2 \frac{f^2}{1-f^2}, \\
D &= -2g\alpha^2 \frac{1+f^2}{1-f^2} + 2g\xi,
\end{aligned} \tag{5.55}
$$

and asymptotic behaviour

$$
\begin{aligned}
f &\sim \ \text{const}\ r^{\pm n}, \quad C \sim r, \quad W_\theta \sim \pm g(\xi - \alpha^2) r^2 \ \text{ for } \ r \to 0, \\
f &\to \sqrt{\frac{\xi - \alpha^2}{\xi + \alpha^2}}, \quad W_\theta \to \frac{n}{4g} \ \text{ for } \ r \to \infty.
\end{aligned} \tag{5.56}
$$

The upper or lower sign apply for positive or negative winding number n, respectively.

The metric for $r \to \infty$ is given by

$$
\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2 \left[1 \mp \frac{1}{2} n(\xi - \alpha^2)\right]^2 \mathrm{d}\theta^2. \tag{5.57}
$$

The asymptotic behaviour is similar to the case of [1]. At $r \to \infty$, the string creates a locally-flat conical metric with a deficit angle proportional to $\xi - \alpha^2$. The energy of the string per unit length can be computed as in [1]. The details of the calculation are given in section 6.2 below: one finds that the only non-vanishing contribution comes from the Gibbons-Hawking surface term [46]

$$
\mu_{\text{string}} = -\int \mathrm{d}\theta\, C' \Big|_{r=\infty} + \int \mathrm{d}\theta\, C' \Big|_{r=0} = \pm \pi n(\xi - \alpha^2) > 0. \tag{5.58}
$$

Note also that the full $N = 2$ supersymmetry is restored asymptotically.

6. $N = 1$ D-term strings with arbitrary Kähler potentials

6.1 Comparison with $N = 1$ supergravity

First we check that the BPS equations derived from the truncated $N = 2$ theory are consistent with the general expressions for $N = 1$ supergravity [47], which in the present form can be found in [48, 15]. The $N = 1$ action, completely determined by the Kähler-potential $\mathcal{K}(\phi, \phi^*)$, the holomorphic function $f_{\alpha\beta}(\phi)$ (no superpotential), the gauging and the FI terms, is[7]

$$
e^{-1}\mathcal{L}_{\text{bos}} = \frac{1}{2}R - g_i{}^j (\nabla_\mu \phi^i)(\nabla^\mu \phi_j) - V_D - \frac{1}{4}(\operatorname{Re} f_{\alpha\beta}) F_{\mu\nu}^\alpha F^{\mu\nu\,\beta} + \frac{1}{8} e^{-1} \varepsilon^{\mu\nu\rho\sigma} (\operatorname{Im} f_{\alpha\beta}) F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta. \tag{6.1}
$$

with covariant derivative given by

$$
\nabla_\mu \phi^i = \partial_\mu \phi^i + g k_\alpha^i\, W_\mu^\alpha. \tag{6.2}
$$

[7] Note that for easy comparison with the $N = 1$ papers, the index i now refers to chiral multiplets, and thus will only take one value in our example: $\phi_i = \phi$ and $\phi^i = \bar{\phi}$. For the fermions: $\chi_i = \chi_L$ and $\chi^i = \chi_R$. On the other hand α now refers to the different vector multiplets.

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The potential consists only of a D-term:

$$V_D = \frac{1}{2}(\mathrm{Re}\, f_{\alpha\beta})D^\alpha D^\beta = \frac{1}{2}(\mathrm{Re}\, f)^{-1\,\alpha\beta}\mathcal{P}_\alpha\mathcal{P}_\beta, \tag{6.3}$$

where

$$\partial_i\mathcal{P}_\alpha(\phi,\phi^*) = -\mathrm{i}g\, k_{\alpha j}g^j{}_i. \tag{6.4}$$

Here, the moment map appears only differentiated, and one can thus add an arbitrary constant, which is the FI term.

The $N=1$ supersymmetry transformations for a bosonic configuration are given by:

$$\begin{aligned}
\delta\psi_{\mu L} &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}(e)\gamma_{ab} + \frac{1}{2}\mathrm{i}A_\mu^B\right)\epsilon_L,\\
\delta\chi_i &= \frac{1}{2}\,\not{\nabla}\phi_i\epsilon_R,\\
\delta\lambda^\alpha &= \frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}^\alpha\epsilon + \frac{1}{2}\mathrm{i}\gamma_5(\mathrm{Re}\, f)^{-1\,\alpha\beta}\mathcal{P}_\beta\epsilon,
\end{aligned} \tag{6.5}$$

with $\epsilon_{L,R} = \frac{1}{2}(1\pm\gamma_5)\epsilon$, and with composite gauge field given by:

$$A_\mu^B = \frac{1}{2}\mathrm{i}\left[(\partial_i\mathcal{K})\partial_\mu\phi^i - (\partial^i\mathcal{K})\partial_\mu\phi_i\right] + W_\mu^\alpha\mathcal{P}_\alpha. \tag{6.6}$$

The comparison with the $N=2$ formulae goes by the substitutions

$$\begin{aligned}
\epsilon_L &= \epsilon^1, & \psi_{\mu L} &= \psi_\mu^1,\\
\lambda_L^\alpha &= -\lambda_2^\beta\mathcal{D}_\beta X^I, & D^\alpha &= -2g\,\mathrm{Im}\,\mathcal{N}^{-1|IJ}\mathcal{P}_J^3,\\
\mathrm{Re}\, f &= -\,\mathrm{Im}\,\mathcal{N}, & \mathcal{P} &= 2g\mathcal{P}^3.
\end{aligned} \tag{6.7}$$

Note that in the second line β on the right-hand side is the index related to coordinates of the special Kähler manifold, while the α on the left-hand side labels the vectors, and corresponds to the index I on the right-hand side.

In the models that we consider we have $\mathrm{Im}\,\mathcal{N}_{IJ} = -\delta_{IJ}$. This implies the relation (3.9). As in the point $z=0$ of the special manifold that we consider, (4.2) leads to

$$\mathcal{D}_z X^1 = \mathrm{e}^{\mathcal{K}/2} = \frac{1}{\sqrt{2}}, \tag{6.8}$$

the $N=1$ gaugino is

$$\lambda_L = -\frac{1}{\sqrt{2}}\lambda_2. \tag{6.9}$$

Therefore, the gravitino and gaugino transformation in (4.16) are in agreement with (6.5).

Identifying Φ with ϕ_i, the kinetic terms are derivable from a Kähler potential

$$\mathcal{K}_{N=1} = -\alpha^2\log(2\,\mathrm{Im}\,\Phi). \tag{6.10}$$

This identifies also (4.17) with (6.6).

The case of one vector and one chiral multiplet with complex field Φ parametrizing the upper half plane $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$ can now be analysed. If the compact isometry (isotropy w.r.t. the base point $\Phi = \mathrm{i}$) $\delta\Phi = 2g(\Phi^2 + 1)$ is gauged and a constant FI term is added, the corresponding D-term is obtained in (3.25). The vacuum manifold is a circle with centre at $(0, \mathrm{i}\xi/\alpha^2)$ and radius $(\sqrt{\xi^2/\alpha^4 - 1})$.

If we now perform the gauging described above, and take the gauge coupling function to be the identity, we see that the BPS equations reduce to the equations (4.16) for ϵ^1 that we get from the truncation, as should be the case.

6.2 Energy of D-term strings with general Kähler target spaces

We determine the energy per unit length for cosmic string configurations in the presence of an arbitrary number of chiral multiplets with a generic Kähler-potential $\mathcal{K}$ and Abelian gauging (i.e. for a generic choice of Killing vector $\delta_\alpha \phi_i = -k_{\alpha i}$.).

Once we take the ansatz for the metric as in (5.1) and we fix the field strength to have only non zero $F^\alpha_{r\theta}$ component, corresponding to a magnetic field in the z direction, we obtain the projector

$$\gamma^{12}\epsilon_L = \mp \mathrm{i}\epsilon_L, \tag{6.11}$$

and the following BPS conditions:

$$\begin{aligned}
&\left(\nabla_r \pm \frac{\mathrm{i}}{C}\nabla_\theta\right)\phi_i = 0,\\
&F^\alpha_{12} \mp D^\alpha = 0,\\
&C'' \pm F^B_{r\theta} = 0,
\end{aligned} \tag{6.12}$$

where $F^B_{\mu\nu} \equiv \partial_\mu A^B_\nu - \partial_\nu A^B_\mu$.

The total energy per unit length is given by:

$$\begin{aligned}
\mu_{\text{string}} = &\int \sqrt{\det g}\, \mathrm{d}r\mathrm{d}\theta \left[g_i{}^j(\nabla_\mu\phi_j)(\nabla^\mu\phi^j) + \frac{1}{4}(\mathrm{Re}\, f_{\alpha\beta})F^\alpha_{\mu\nu}\, F^{\beta\mu\nu} + \frac{1}{2}D^2 - \frac{1}{2}R\right]\\
&+ \left(\int \mathrm{d}\theta\sqrt{\det h}\; K\bigg|_{r=\infty} - \int \mathrm{d}\theta\sqrt{\det h}\; K\bigg|_{r=0}\right),
\end{aligned} \tag{6.13}$$

where the sums over μ, ν run only over r, θ. The quantity K is the Gaussian curvature at the boundaries (on which the metric is h), which are at $r = \infty$ and $r = 0$. For the metric (5.1) we have:

$$\sqrt{\det g} = C(r), \qquad \sqrt{\det g}\, R = -2C'', \qquad \sqrt{\det h}\, K = -C'. \tag{6.14}$$

We now argue that the energy (6.13) can always be obtained in terms of the BPS equations (6.12). Indeed, consider the combination

$$\begin{aligned}
\mu_{\text{string}} = \int \mathrm{d}r\mathrm{d}\theta\, C(r)\Big[&g_i{}^j(\nabla_r \pm \mathrm{i}C^{-1}\nabla_\theta)\phi^i(\nabla_r \mp \mathrm{i}C^{-1}\nabla_\theta)\phi^j\\
&+ \frac{1}{2}(F^\alpha_{12} \mp D^\alpha)(\mathrm{Re}\, f_{\alpha\beta})(F^\beta_{12} \mp D^\beta)\Big]
\end{aligned}$$

$$
+\int \mathrm{d}r\mathrm{d}\theta\ \left[C'' \pm F^B_{r\theta}\right] - \int \mathrm{d}\theta\, C'\Big|_{r=\infty} + \int \mathrm{d}\theta\, C'\Big|_{r=0}. \tag{6.15}
$$

Using (6.4), we can derive from (6.6):

$$
F^B_{\mu\nu} = 2\mathrm{i}g_i{}^j\partial_{[\mu}\phi_j\partial_{\nu]}\phi^i + F^\alpha_{\mu\nu}\mathcal{P}_\alpha - 2\mathrm{i}W^\alpha_{[\mu}\left[(\partial_{\nu]}\phi^i k_{\alpha j} + \partial_{\nu]}\phi_j k_\alpha{}^i)g_i{}^j\right]. \tag{6.16}
$$

Now one can, after some calculation, reconstruct the kinetic terms for the gauge field and the scalars, and the potential for the scalars in (6.13).

The conclusion is that any U(1) gauging of a compact isometry on any Kähler target space can give rise to a cylindrically symmetric BPS string with mass per unit length given by the Gibbons-Hawking surface term (if the BPS equations can be solved[8]). We use this result when we give the energy density for our string solution of section 5. One can also check explicitly that the $N = 1$ equations of motion hold in this case. The analysis also applies to the semilocal string solutions discussed in [24, 25] (see also [26, 49]), and the axionic D-term strings discussed in [11], where a non trivial Kähler potential was also considered.

However suggestive, this Bogomol'nyi-type argument cannot be used directly to conclude that the solutions are stable, as non-axisymmetric or z-dependent perturbations might destabilize the strings. But we expect the Bogomol'nyi bound can be generalized to non-axisymmetric and multi-vortex configurations along the lines of [50].

Also, when the BPS strings are non-topological, the presence of cylindrically symmetric zero modes that make the magnetic field spread can prevent the strings from forming in a cosmological context (see [51 – 53, 25] for a discussion of this point in the context of BPS semilocal strings).

7. Discussion

In this work, we have studied the embedding of four-dimensional $N = 1$ D-term string solutions in $N = 2$ supergravity models. We have shown how an $N = 2$ action with U(1) gauging can be reduced by a consistent truncation to an $N = 1$ action with Fayet-Iliopoulos term and vanishing superpotential. Especially important in the construction are the choices of gauging and special geometry.

Alternatively, one can use this information to devise an ansatz which allows the full $N = 2$ BPS equations to be solved explicitly. In this way the half-BPS $N = 1$ D-term strings are promoted to half-BPS $N = 2$ strings.

The reduced $N = 1$ action has a charged scalar parametrizing a non trivial Kähler-Hodge target space. We have shown how to solve the resulting $N = 1$ BPS-equations for this system, along the lines of [1], and demonstrated that the half-BPS solution is also half-BPS in the $N = 2$ context. A full stability analysis of these solutions has not been presented and is a very important issue. Another interesting related question is the possible interactions of the strings with black holes.

[8]An explicit numerical solution of BPS equations in a similar situation has been given in [10, 11].

The way to obtain arbitrary FI constants in $N = 1$ by consistent truncations can be useful in a wider context than that of string solutions, while the results for $N = 1$ theories with arbitrary Kähler-Hodge geometries can be of interest in the current research concerning cosmic string solutions in string theory.

We have presented our results for both normal quaternionic manifolds of dimension one. The generalization to normal higher dimensional quaternionic manifolds is straightforward since any normal quaternionic manifold admits one of the quaternionic manifolds of dimension one as a completely geodesic submanifold with a compatible quaternionic structure.

The issue of Kähler anomalies in gauged $N = 1$ supergravity with a non-trivial Kähler manifold was recently brought up in [54]. The possible anomalies discussed in that paper are due to the U(1) R-symmetry group. The setting in this paper of $N = 2$ theories avoids such problems. The R-symmetry group of $N = 2$, $D = 4$ supergravity is $\mathrm{U}(1) \times \mathrm{SU}(2)$. The gauging that we consider is an Abelian gauging that does not act on the scalars of the vector multiplets, but only on the hyperscalars. As the hyperscalars are inert under the U(1) factor of the R-symmetry group, we are only concerned with the SU(2) factor. The hyperini transform under $\mathrm{Sp}(m, \mathbb{R})$, which is a real representation and therefore free of anomalies. The gaugini and the gravitini are charged under the SU(2) of the R-symmetry of $N = 2$ and transform as a doublet. But SU(2) is not an anomalous group for local gauging so the gaugini and gravitini will not bring about any anomalies. We can thus conclude that our gauging is going to be free of anomalies as the fermions of the theory transform in an anomaly free representation and the U(1) of the R symmetry (coming from special geometry) is not gauged. Note that this explanation is quite general and can apply to any Abelian gauging in $N = 2$ supergravity. From this point of view it is safer to consider the string solution as living in $N = 2$ although the solution involves only fields related to an $N = 1$ subsector.

BPS solitons and defects can usually be coupled to gravity without losing their BPS character. This is true for $N = 1$ D-term strings, as [1] showed. But in $N = 2$ global supersymmetry there are also half-BPS cosmic strings [40], and the question arises as to what is the fate of these solutions when coupled to (super)gravity. One would naively expect to be able to find the corresponding half-BPS solutions in $N = 2$ supergravity and moreover they should not be too different from the $N = 1$ ones, because in the absence of gravity they are equivalent. Until now, the stumbling block in giving these solutions an $N = 2$ interpretation was the difficulty in constructing constant Fayet-Iliopoulos terms in $N = 2$ supergravity and it is reassuring that this difficulty can be circumvented.

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A. Notation

Our metric is mostly +, and we use the (+++) conventions in the Misner-Thorne-Wheeler classification [55] scheme, such that compact spaces have positive scalar curvature, and covariant derivatives on fermions have the form

$$\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}. \tag{A.1}$$

Antisymmetrization is done with weight 1, see (2.2). We use indices μ and a for local and tangent spacetime. The $N = 2$ extension index is $i = 1, 2$. This is related to SU(2) vectors, labelled by $x = 1, 2, 3$, using the Pauli matrices:

$$A_i{}^j \equiv \mathrm{i}A^x\left(\sigma^x\right)_i{}^j, \qquad \text{or} \qquad A^x = -\frac{1}{2}\mathrm{i}\,\mathrm{tr}\,\sigma^x A. \tag{A.2}$$

Lowering and raising SU(2) indices is done using the ε symbol, in northwest-southeast (NW-SE) conventions,

$$A^i = \varepsilon^{ij}A_j, \qquad A_i = A^j\varepsilon_{ji}. \tag{A.3}$$

γ_5 and the Levi-Civita symbol are normalized as

$$\varepsilon_{0123} = 1, \qquad \varepsilon^{0123} = -1, \qquad \gamma_5 \equiv \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3. \tag{A.4}$$

Quaternions are written as 2×2 complex matrices using

$$q = q^0\sigma^0 + \mathrm{i}q^x\sigma^x. \tag{A.5}$$

Hermitian conjugation and the anti-Hermitian part are denoted as

$$\bar{q} = q^0\sigma^0 - \mathrm{i}q^x\sigma^x, \qquad \vec{q} = \mathrm{i}q^x\sigma^x. \tag{A.6}$$

Translations between forms and components are done with factors and signs as in

$$J = \frac{1}{2}J_{XY}\mathrm{d}q^X\mathrm{d}q^Y, \qquad \mathrm{d}(A_X\mathrm{d}q^X) = \partial_Y A_X\mathrm{d}q^Y\mathrm{d}q^X. \tag{A.7}$$

All the properties and conventions on hypermultiplets that we follow can be found in [56], especially in appendix B, except from the change of notation that we now indicate a 3-vector with indices x rather than α in that paper. The spinors of hypermultiplets are labelled by $A = 1, \ldots, 2n_H$. These are Sp(n_H) indices, which we will sometimes split in further SU(2) indices $i = 1, 2$ and vector indices $t, s = 1, \ldots, n_H$. The former index will be put in opposite up/down position, such that e.g. the 1-form vielbein f^{iA} becomes with $A = (tj)$ for every value of t a 2×2 matrix $f^t{}_j{}^i$. The symplectic metric is then split as

$$\mathbb{C}_{AB} = \varepsilon^{ij} \otimes \mathbb{1}_{ts} \qquad \text{for } A = (ti),\ B = (sj). \tag{A.8}$$

The reality condition (with complex conjugation denoted as $*$),

$$(f^{iA})^* = f^{jB} \varepsilon_{ji} \mathbb{C}_{BA}, \tag{A.9}$$

translates then to the property that f^t as a 2×2 matrix ('quaternion') satisfies

$$(f^t)^* = \sigma_2 f^t \sigma_2. \tag{A.10}$$

This is satisfied for quaternions of the form (A.5) with q^0 and q^x real. We have then

$$\begin{aligned} &g_{XY} f^Y_{iA} = (\bar{f}^t_X)_i{}^j \quad \text{with } A = (tj), \qquad g_{XY} = \operatorname{tr}(f^t_X \bar{f}^t_Y), \\ &(J^x)_{XY} = (J^x)_X{}^Z g_{ZY} = -\mathrm{i} \operatorname{tr}(f^t_X \sigma^x \bar{f}^t_Y), \end{aligned} \tag{A.11}$$

or the 2-form hypercomplex structure as a quaternion is

$$\vec{J} = -\bar{f}^t \wedge f^t. \tag{A.12}$$

B. Parametrization of coset spaces

We encounter in this paper two 1-dimensional quaternionic-Kähler coset spaces. Both can be expressed as submanifolds of the 2-dimensional quaternionic-Kähler manifold $\frac{\mathrm{Sp}(2,1)}{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}$. We give here the parametrizations that we use. Though the 1-quaternion coset spaces can be used without reference to the 2-quaternion one, we will start by the parametrization of the latter, and determine parametrizations of the others as truncations thereof.

B.1 Parametrization of the 2-dimensional projective quaternionic space

We shall consider the quaternionic-Kähler manifold of quaternionic dimension 2:

$$\frac{\mathrm{Sp}(2,1)}{\mathrm{Sp}(2) \times \mathrm{Sp}(1)} \simeq \frac{\mathrm{USp}(4,2)}{\mathrm{USp}(4) \times \mathrm{USp}(2)}. \tag{B.1}$$

The algebra of the isometry group, $\mathfrak{sp}(2,1)$ can be defined as the set of matrices over the quaternions $\mathbb{H}$ that preserve a metric of signature $(+,+,-)$. We take this metric in the form

$$\mu = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \tag{B.2}$$

where each entry is a quaternion, or 2×2 complex matrix. The elements M of $\mathfrak{sp}(2,1)$ are those 3×3 matrices with entries in $\mathbb{H}$ that satisfy

$$\mu M^\dagger \mu = -M. \tag{B.3}$$

The general form of an element of $\mathfrak{sp}(2,1)$ is then

$$M = \begin{pmatrix} a & \frac{1}{2}(\bar{e}+\bar{f}) & -\frac{1}{2}(\vec{b}+\vec{c}) \\ \frac{1}{2}(e-f) & \vec{p} & -\frac{1}{2}(e+f) \\ \frac{1}{2}(\vec{b}-\vec{c}) & \frac{1}{2}(\bar{f}-\bar{e}) & -\bar{a} \end{pmatrix}, \tag{B.4}$$

where $a = a_0 + \vec{a}$, $e = e_0 + \vec{e}$ and $f = f_0 + \vec{f}$ are generic quaternions and $\vec{c}$, $\vec{b}$ and $\vec{p}$ are pure anti-Hermitian quaternions (with vanishing Hermitian part).[9]

The Lie algebra of $\mathfrak{sp}(2,1)$ can be split into a compact (anti-Hermitian) and non-compact (Hermitian) part :

$$M_H = \begin{pmatrix} \vec{a} & \frac{1}{2}\bar{f} & -\frac{1}{2}\vec{c} \\ -\frac{1}{2}f & \vec{p} & -\frac{1}{2}f \\ -\frac{1}{2}\vec{c} & \frac{1}{2}\bar{f} & \vec{a} \end{pmatrix}, \quad M_{G/H} = \begin{pmatrix} a_0\mathbb{1} & \frac{1}{2}\bar{e} & -\frac{1}{2}\vec{b} \\ \frac{1}{2}e & 0 & -\frac{1}{2}e \\ \frac{1}{2}\vec{b} & -\frac{1}{2}\bar{e} & -a_0\mathbb{1} \end{pmatrix}. \tag{B.5}$$

The H part of the generator can be decomposed into its subalgebras[10] :

$$M_{\mathfrak{sp}(1)} == \begin{pmatrix} \vec{u} & 0 & -\vec{u} \\ 0 & 0 & 0 \\ -\vec{u} & 0 & \vec{u} \end{pmatrix}, \quad M_{\mathfrak{sp}(2)} = \begin{pmatrix} \vec{v} & \frac{1}{2}\bar{f} & \vec{v} \\ -\frac{1}{2}f & \vec{p} & -\frac{1}{2}f \\ \vec{v} & \frac{1}{2}\bar{f} & \vec{v} \end{pmatrix}. \tag{B.6}$$

$M_{\mathfrak{sp}(1)}$ commutes with $M_{\mathfrak{sp}(2)}$ and the latter contains two commuting $\mathfrak{su}(2)$ parameterized by $\vec{p}$ and $\vec{v}$:

$$M_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)\subset\mathfrak{sp}(2)} = \begin{pmatrix} \vec{v} & 0 & \vec{v} \\ 0 & \vec{p} & 0 \\ \vec{v} & 0 & \vec{v} \end{pmatrix}. \tag{B.7}$$

We see that the compact subalgebra of $\mathfrak{sp}(2,1)$ contains three commuting $\mathfrak{su}(2)$. The first $\mathfrak{su}(2)$ factor, $M_{\mathfrak{sp}(1)}$, corresponds to the R-symmetry whereas the $\mathfrak{su}(2)_{\vec{p}} \subset \mathfrak{sp}(2)$ contains the compact U(1) for the string.

The solvable gauge of the coset manifold is obtained by adding to $M_{G/H}$ an element of M_H (with $\vec{c} = \vec{b}$, $f = e$ and $\vec{a} = \vec{p} = 0$) so that the result is an upper triangular matrix:

$$M_{\text{Solvable}} = \begin{pmatrix} a_0\mathbb{1} & \bar{e} & -\vec{b} \\ 0 & 0 & -e \\ 0 & 0 & -a_0\mathbb{1} \end{pmatrix}. \tag{B.8}$$

B.2 Solvable coordinates and metric of $\frac{\mathrm{Sp}(2,1)}{\mathrm{Sp}(2)\,\mathrm{Sp}(1)}$

We parametrize the coset elements by

$$L = \mathrm{e}^N \cdot \mathrm{e}^H, \tag{B.9}$$

[9] The identification $\mathfrak{sp}(2,1) \simeq \mathfrak{usp}(4,2)$ is obtained once we take the matrices $-\mathrm{i}\vec{\sigma}$ for the imaginary quaternions.

[10] It is related to the previous expression of M_H by taking $\vec{u} = \frac{1}{2}\vec{a} + \frac{1}{4}\vec{c}$ and $\vec{v} = \frac{1}{2}\vec{a} - \frac{1}{4}\vec{c}$.

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where

$$N = N_e + N_b = \underbrace{\begin{pmatrix} 0 & \bar{e} & 0 \\ 0 & 0 & -e \\ 0 & 0 & 0 \end{pmatrix}}_{N_e} + \underbrace{\begin{pmatrix} 0 & 0 & -\vec{b} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{N_b}, \quad H = \frac{1}{2}\begin{pmatrix} h\mathbb{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -h\mathbb{1} \end{pmatrix}. \tag{B.10}$$

The coordinates q^X are thus the real h, the 3 real coordinates of $\vec{b}$ and the 4 real parts of the quaternion e. This leads to

$$L = \begin{pmatrix} \mathrm{e}^{\frac{1}{2}h}\mathbb{1} & \bar{e} & -\mathrm{e}^{-\frac{1}{2}h}(\vec{b} + \frac{\bar{e}e}{2}) \\ 0 & \mathbb{1} & -\mathrm{e}^{-\frac{1}{2}h}e \\ 0 & 0 & \mathrm{e}^{-\frac{1}{2}h}\mathbb{1} \end{pmatrix}. \tag{B.11}$$

This leads to the algebra element

$$L^{-1}\mathrm{d}L = \begin{pmatrix} \frac{B_0}{2} & \frac{\bar{E}}{\sqrt{2}} & -\vec{B} \\ 0 & 0 & -\frac{E}{\sqrt{2}} \\ 0 & 0 & -\frac{B_0}{2} \end{pmatrix}, \tag{B.12}$$

where

$$B = B_0\mathbb{1} + \vec{B} = \mathrm{d}h\mathbb{1} + \mathrm{e}^{-h}\left[\mathrm{d}\vec{b} - \frac{1}{2}(\bar{e}\mathrm{d}e - \mathrm{d}\bar{e}e)\right], \quad E = \sqrt{2}\,\mathrm{e}^{-\frac{1}{2}h}\mathrm{d}e, \tag{B.13}$$

or in real components

$$B_0 = \mathrm{d}h, \qquad B^x = \mathrm{e}^{-h}\left(\mathrm{d}b^x + e^x\mathrm{d}e^0 - e^0\mathrm{d}e^x - \varepsilon^{xyz}e^y\mathrm{d}e^z\right). \tag{B.14}$$

The algebra element can be split in the coset part and the part in H. The first one is the Hermitian part:

$$(L^{-1}\mathrm{d}L)_{G/H} = \frac{1}{2}\begin{pmatrix} B_0 & \frac{\bar{E}}{\sqrt{2}} & -\vec{B} \\ \frac{E}{\sqrt{2}} & 0 & -\frac{E}{\sqrt{2}} \\ \vec{B} & -\frac{\bar{E}}{\sqrt{2}} & -B_0. \end{pmatrix}. \tag{B.15}$$

The part in H is the anti-Hermitian part, which can be split in the $\mathfrak{sp}(1)$ and $\mathfrak{sp}(2)$ part:

$$\begin{aligned}
(L^{-1}\mathrm{d}L)_H &= \frac{1}{2}\begin{pmatrix} 0 & \frac{\bar{E}}{\sqrt{2}} & -\vec{B} \\ -\frac{E}{\sqrt{2}} & 0 & -\frac{E}{\sqrt{2}} \\ -\vec{B} & \frac{\bar{E}}{\sqrt{2}} & 0 \end{pmatrix} = (L^{-1}\mathrm{d}L)_{\mathfrak{sp}(1)} + (L^{-1}\mathrm{d}L)_{\mathfrak{sp}(2)}, \\
(L^{-1}\mathrm{d}L)_{\mathfrak{sp}(1)} &= \frac{1}{4}\begin{pmatrix} \vec{B} & 0 & -\vec{B} \\ 0 & 0 & 0 \\ -\vec{B} & 0 & \vec{B} \end{pmatrix}, \\
(L^{-1}\mathrm{d}L)_{\mathfrak{sp}(2)} &= \begin{pmatrix} -\frac{1}{4}\vec{B} & \frac{\bar{E}}{\sqrt{2}} & -\frac{1}{4}\vec{B} \\ -\frac{E}{\sqrt{2}} & 0 & -\frac{E}{\sqrt{2}} \\ -\frac{1}{4}\vec{B} & \frac{\bar{E}}{\sqrt{2}} & -\frac{1}{4}\vec{B} \end{pmatrix}.
\end{aligned} \tag{B.16}$$

The metric is defined as

$$\mathrm{d}s^2 = g_{XY}\mathrm{d}q^X\mathrm{d}q^Y = \mathrm{Tr}\left[(L^{-1}\mathrm{d}L)_{G/H}\cdot(L^{-1}\mathrm{d}L)_{G/H}\right] = \frac{1}{2}\,\mathrm{tr}(B\bar{B}+E\bar{E}), \tag{B.17}$$

where Tr stands for a trace over the 6×6 matrix and tr for a trace over the 2×2 matrix. We will comment on the normalization of this metric below. Its value is

$$\mathrm{d}s^2 = (\mathrm{d}h)^2 + (B^1)^2 + (B^2)^2 + (B^3)^2 + 2\mathrm{e}^{-h}\left[(\mathrm{d}e^0)^2 + (\mathrm{d}e^1)^2 + (\mathrm{d}e^2)^2 + (\mathrm{d}e^3)^2\right]. \tag{B.18}$$

The vielbeins, as 1-forms and quaternions as explained above, can be taken to be

$$f^1 = \frac{1}{\sqrt{2}}B, \qquad f^2 = \frac{1}{\sqrt{2}}E. \tag{B.19}$$

These lead to (B.17) and to the hypercomplex form ($\wedge$ symbols understood)

$$\vec{J} = -\frac{1}{2}\left(\bar{B}\,B + \bar{E}\,E\right), \qquad \text{or} \qquad J^x = -B_0B^x - E_0E^x - \frac{1}{2}\varepsilon^{xyz}\left(B^yB^z + E^yE^z\right). \tag{B.20}$$

Using the differentials

$$\begin{aligned} &\mathrm{d}B = -B_0B - \frac{1}{2}\bar{E}\,E, \qquad \mathrm{d}E = -\frac{1}{2}B_0E,\\ &\text{or} \qquad \mathrm{d}B^x = -B_0B^x - E_0E^x - \frac{1}{2}\varepsilon^{xyz}E^yE^z, \end{aligned} \tag{B.21}$$

we obtain

$$\mathrm{d}J^x + 2\varepsilon^{xyz}\omega^yJ^z = 0, \tag{B.22}$$

for

$$\omega^x = -\frac{1}{2}B^x. \tag{B.23}$$

We find then that (2.15) is satisfied for $\nu = -1$. The value that we get here for ν depends on the normalization of the metric. Multiplying the metric by an arbitrary $-\nu^{-1}$, would lead to (2.15) with this arbitrary value of ν. In the supergravity context, $\nu = -\kappa^2$, where κ is the gravitational coupling constant, which we have put equal to 1.

B.3 The projective quaternionic manifold $\frac{\mathrm{Sp}(1,1)}{\mathrm{Sp}(1)\mathrm{Sp}(1)}$

The quaternionic-Kähler manifold of quaternionic dimension 1

$$\frac{\mathrm{Sp}(1,1)}{\mathrm{Sp}(1)\mathrm{Sp}(1)} \tag{B.24}$$

can be seen as a submanifold of $\frac{\mathrm{Sp}(2,1)}{\mathrm{Sp}(2)\mathrm{Sp}(1)}$ by taking $E=0$. The metric is then

$$\mathrm{d}s^2 = (\mathrm{d}h)^2 + \mathrm{e}^{-2h}\left[(\mathrm{d}b^1)^2 + (\mathrm{d}b^2)^2 + (\mathrm{d}b^3)^2\right]. \tag{B.25}$$

The vielbein and hypercomplex forms are simply obtained from the previous section by putting $e = E = 0$. E.g.

$$\mathcal{R}^x = \frac{1}{2}\mathrm{e}^{-h}\mathrm{d}h\wedge\mathrm{d}b^x + \frac{1}{4}\mathrm{e}^{-2h}\varepsilon^{xyz}\mathrm{d}b^y\wedge\mathrm{d}b^z, \quad x,y,z = 1,2,3, \tag{B.26}$$

and can be obtained as the curvature of the connection

$$\omega^x = -\frac{1}{2}\mathrm{e}^{-h}\mathrm{d}b^x. \tag{B.27}$$

B.4 The normal quaternionic manifold $\frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)}$

The quaternionic-Kähler manifold of dimension one

$$\frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)} \tag{B.28}$$

can be defined as a submanifold of $\frac{\mathrm{Sp}(1,1)}{\mathrm{Sp}(1)\mathrm{Sp}(1)}$ in many different ways. Here we will consider the choice

$$b^1 = b^2 = e^0 = e^3 = 0. \tag{B.29}$$

The metric can be obtained from reducing (B.18). However, in order to respect the $\nu = -1$ normalization as explained at the end of section B.2, we need here another overall factor. We have

$$\mathrm{d}s^2 = \frac{1}{2}\mathrm{d}h^2 + \frac{1}{2}\mathrm{e}^{-2h}(\mathrm{d}b^3 - e^1\mathrm{d}e^2 + e^2\mathrm{d}e^1)^2 + \mathrm{e}^{-h}\left[(\mathrm{d}e^1)^2 + (\mathrm{d}e^2)^2\right]. \tag{B.30}$$

We can again obtain all expressions from those of $\frac{\mathrm{Sp}(1,1)}{\mathrm{Sp}(1)\mathrm{Sp}(1)}$, where B has only the B_0 and B^3 components, and E has only E^1 and E^2. The vielbein is

$$f = \frac{1}{2}(\bar{B} + \bar{E}) = \begin{pmatrix} v & -t^* \\ t & v^* \end{pmatrix}, \tag{B.31}$$

where

$$v = \frac{1}{2}\left[\mathrm{d}h - \mathrm{i}\mathrm{e}^{-h}(\mathrm{d}b^3 - e^1\mathrm{d}e^2 + e^2\mathrm{d}e^1)\right], \quad t = \frac{1}{\sqrt{2}}\mathrm{e}^{-h/2}(\mathrm{d}e^2 - \mathrm{i}\mathrm{d}e^1). \tag{B.32}$$

The quaternionic form is

$$\mathcal{R} = \mathrm{i}\mathcal{R}^x(\sigma^x) = \frac{1}{2}\bar{f}\wedge f = \frac{1}{2}\begin{pmatrix} v^*\wedge v + t^*\wedge t & 2\,t^*\wedge v^* \\ 2v\wedge t & v\wedge v^* + t\wedge t^* \end{pmatrix}, \tag{B.33}$$

which gives

$$\begin{aligned}
\mathcal{R}^1 &= \frac{1}{2\sqrt{2}}\left[\mathrm{e}^{-h/2}\mathrm{d}e^1\wedge\mathrm{d}h + \mathrm{e}^{-3h/2}\mathrm{d}e^2\wedge(\mathrm{d}b^3 + e^2\mathrm{d}e^1)\right],\\
\mathcal{R}^2 &= \frac{1}{2\sqrt{2}}\left[\mathrm{e}^{-h/2}\mathrm{d}e^2\wedge\mathrm{d}h - \mathrm{e}^{-3h/2}\mathrm{d}e^1\wedge(\mathrm{d}b^3 - e^1\mathrm{d}e^2)\right],\\
\mathcal{R}^3 &= -\frac{1}{4}\mathrm{e}^{-h}\mathrm{d}h\wedge(\mathrm{d}b^3 + e^2\mathrm{d}e^1 - e^1\mathrm{d}e^2) + \frac{1}{2}\mathrm{e}^{-h}\mathrm{d}e^1\wedge\mathrm{d}e^2,
\end{aligned} \tag{B.34}$$

and is the curvature of the connection

$$\omega^1 = \frac{1}{\sqrt{2}}\mathrm{e}^{-h/2}\mathrm{d}e^1, \quad \omega^2 = \frac{1}{\sqrt{2}}\mathrm{e}^{-h/2}\mathrm{d}e^2, \quad \omega^3 = \frac{1}{4}\mathrm{e}^{-h}(\mathrm{d}b^3 + e^2\mathrm{d}e^1 - e^1\mathrm{d}e^2). \tag{B.35}$$

This parametrization of the manifold can also be found in [39] with the following replacements:

$$h = \log V, \qquad b^3 = -\sigma, \qquad e^1 = -\sqrt{2}\tau, \qquad e^2 = \sqrt{2}\theta. \tag{B.36}$$

The chosen vielbeins differ by a multiplication by $\mathrm{i}\sigma_2$ on the side of the indices A, B. That does not change the complex structures. Note, however, that our conventions differ by the two signs in (A.7) such that the 2-forms differ by a sign.

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